Université de Picardie Jules Verne

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Analyse

Niveau 1 : Unité fondamentale (Mias)

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Exercice 1

Soit $n \in \mathbb{N}$ et $x_0 \in \mathbb{R}$.

1. Montrer que $\lim_{x \to x_0} x^n = (x_0)^n$.

   (On rappelle que $(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k$)

2. Montrer que $\lim_{x \to x_0} \alpha x^n = \alpha (x_0)^n \quad \forall \alpha \in \mathbb{R}$.

3. En déduire que $\lim_{x \to x_0} P(x) = P(x_0)$ pour tout polynôme P càd $\forall P \in \mathbb{R}[X]$.

4. En déduire que $\lim_{x \to x_0} R(x) = R(x_0)$ pour toute fraction rationnelle R càd $\forall R \in \mathbb{R}(X)$ telle que $x_0 \in D_R$.

Correction 1

1. On a $\lim_{x \to x_0} x^n = (x_0)^n \Leftrightarrow \lim_{h \to 0} (x_0+h)^n = (x_0)^n$.

   $(x_0+h)^n - (x_0)^n = \sum_{k=0}^{n} C_n^k x_0^{n-k} h^k - (x_0)^n = \sum_{k=1}^{n} C_n^k x_0^{n-k} h^k$

   Or $\lim_{h \to 0} \alpha h^p = 0 \quad \forall \alpha \in \mathbb{R}$ et $\forall p \in \mathbb{N}^*$. Donc $\lim_{h \to 0} x_0^i h^j = 0 \quad \forall i, j \in \mathbb{N}^*$.

   On a $\lim_{h \to 0} \sum_{k=1}^{n} C_n^k x_0^{n-k} h^k = 0$ et $\lim_{h \to 0} (x_0+h)^n - (x_0)^n = 0$ càd $\lim_{h \to 0} (x_0+h)^n = (x_0)^n$

2. Puisque $\lim_{x \to x_0} \alpha \, \exists$ et $\lim_{x \to x_0} x^n \, \exists$, on a $\lim_{x \to x_0} \alpha x^n = \lim_{x \to x_0} \alpha \times \lim_{x \to x_0} x^n = \alpha (x_0)^n$.

3. Soit $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p$.
   De même chacune des limites existe et est finie, on a donc :
   $\lim_{x \to x_0} P(x) = \lim_{x \to x_0} a_0 + \lim_{x \to x_0} a_1 x + \lim_{x \to x_0} a_2 x^2 + \cdots + \lim_{x \to x_0} a_p x^p$.
   $\lim_{x \to x_0} P(x) = a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_p x_0^p = P(x_0)$.

4. Soit $R(x) = \dfrac{P(x)}{Q(x)}$ avec $P, Q \in R[X]$.

   On a $\lim_{x \to x_0} P(x) = P(x_0) \in \mathbb{R}$ et $\lim_{x \to x_0} Q(x) = Q(x_0) \in \mathbb{R}^*$. Donc $\lim_{x \to x_0} R(x) = \dfrac{P(x_0)}{Q(x_0)} = R(x_0)$.

Exercice 2

Calculer $\lim\limits_{x \to 4} \dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$.

Correction 2

$$\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \frac{(3-\sqrt{5+x})(1+\sqrt{5-x})(3+\sqrt{5+x})}{(1-\sqrt{5-x})(1+\sqrt{5-x})(3+\sqrt{5+x})} = \frac{(4-x)(1+\sqrt{5-x})}{(x-4)(3+\sqrt{5+x})} = -\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}.$$

D'où $\lim\limits_{x \to 4} \dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = -\dfrac{1}{3}$.

Exercice 3

Soient f et g définies sur $\mathbb{R}$ par $f(x) = x^3 + 5x^2 + 6x$ et $g(x) = x^2 + 2x$.

1. (a) A-t-on $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} g(x)$?

   (b) A-t-on $f \underset{+\infty}{\sim} g$?

2. (a) A-t-on $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x)$?

   (b) A-t-on $f \underset{0}{\sim} g$?

Correction 3

Rappel : si $\lim\limits_{x\to a} f(x) = l$ et $\lim\limits_{x\to a} g(x) = l$ avec l fini et non nul alors $f \underset{a}{\sim} g$.

1. (a) $\lim\limits_{x \to +\infty} x^3 = +\infty$, $\lim\limits_{x \to +\infty} 5x^2 = +\infty$ et $\lim\limits_{x \to +\infty} 6x = +\infty$.
   Donc $\lim\limits_{x \to +\infty} f(x) = +\infty$.
   $\lim\limits_{x \to +\infty} x^2 = +\infty$ et $\lim\limits_{x \to +\infty} 2x = +\infty$.
   Donc $\lim\limits_{x \to +\infty} g(x) = +\infty$.

   (b) $\dfrac{f(x)}{g(x)} = \dfrac{x(x+2)(x+3)}{x(x+2)} = x+3$ et $\lim\limits_{x \to +\infty} x+3 = +\infty$.
   Donc $f \not\sim g$ en $+\infty$.

2. (a) $\lim\limits_{x \to 0} f(x) = 0 = \lim\limits_{x \to 0} g(x)$.

   (b) $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = 3$. Donc $f \not\sim g$ en 0.

Exercice 4

1. En admettant que $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, démontrer que $1 - \cos x \underset{0}{\sim} \dfrac{x^2}{2}$.

2. Déterminer $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^4 + x^2}$.

3. Déterminer $\lim\limits_{x \to 0} \dfrac{(1-\cos x)^2}{x^4 + x^2}$.

4. Déterminer $\lim\limits_{x \to +\infty} \dfrac{(1-\cos x)^2}{x^4 + x^2}$.

Correction 4

1. $\cos 2x = 1 - 2\sin^2 x$ donc $\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$ et $1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$.
   $\sin x \underset{0}{\sim} x$ donc $\sin\left(\frac{x}{2}\right) \underset{0}{\sim} \frac{x}{2}$ et $2\sin^2\left(\frac{x}{2}\right) \underset{0}{\sim} \frac{x^2}{2}$.
2. $x^4 + x^2 \underset{0}{\sim} x^2$ donc $\dfrac{1-\cos x}{x^4+x^2} \underset{0}{\sim} \dfrac{1}{2}$ et $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^4+x^2} = \dfrac{1}{2}$.
3. $(1-\cos x)^2 \underset{0}{\sim} \dfrac{x^4}{4}$ donc $\dfrac{(1-\cos x)^2}{x^4+x^2} \underset{0}{\sim} \dfrac{x^2}{4}$ et $\lim\limits_{x \to 0} \dfrac{(1-\cos x)^2}{x^4+x^2} = 0$
4. $0 \leq (1-\cos x)^2 \leq 4$ donc $\lim\limits_{x \to +\infty} \dfrac{(1-\cos x)^2}{x^4+x^2} = 0$

Exercice 5

1. Montrer que $\sqrt{x^3+x} \underset{+\infty}{\sim} x\sqrt{x}$.
2. Déterminer $\lim\limits_{x \to +\infty} \cos\left(\dfrac{x+1}{\sqrt{x^3+x}}\right)$.

Correction 5

1. $x^3 + x \underset{+\infty}{\sim} x^3$ donc $\sqrt{x^3+x} \underset{+\infty}{\sim} \sqrt{x^3} = x\sqrt{x}$.
2. $x+1 \underset{+\infty}{\sim} x$ et $\sqrt{x^3+x} \underset{+\infty}{\sim} \sqrt{x^3} = x\sqrt{x}$. Donc $\dfrac{x+1}{\sqrt{x^3+x}} \underset{+\infty}{\sim} \dfrac{1}{\sqrt{x}}$ et $\lim\limits_{x \to +\infty} \dfrac{x+1}{\sqrt{x^3+x}} = 0$.
   D'où $\lim\limits_{x \to +\infty} \cos\left(\dfrac{x+1}{\sqrt{x^3+x}}\right) = 1$.

Exercice 6

Calculer $\lim\limits_{x \to +\infty} (\sqrt{x^2-5x+6} - x)$.

Correction 6

$\sqrt{x^2-5x+6} - x = \dfrac{x^2-5x+6-x^2}{\sqrt{x^2-5x+6}+x}$.

On a $\sqrt{x^2-5x+6} + x = x\left(\sqrt{1 - \dfrac{5}{x} + \dfrac{6}{x^2}} + 1\right)$.

Or $\lim\limits_{x \to +\infty} \sqrt{1 - \dfrac{5}{x} + \dfrac{6}{x^2}} + 1 = 2$, donc $\lim\limits_{x \to +\infty} \dfrac{\sqrt{x^2-5x+6}+x}{2x} = 1$ càd $\sqrt{x^2-5x+6} + x \underset{+\infty}{\sim} 2x$.

D'où $-5x + 6 \underset{+\infty}{\sim} -5x$ et $\lim\limits_{x \to +\infty} \sqrt{x^2-5x+6} - x = -\dfrac{5}{2}$.

Exercice 7

Calculer $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\sin 2x}$.

Correction 7

$\sin 5x \underset{0}{\sim} 5x$ et $\sin 2x \underset{0}{\sim} 2x$. Donc $\dfrac{\sin 5x}{\sin 2x} \underset{0}{\sim} \dfrac{5x}{2x} \underset{0}{\sim} \dfrac{5}{2}$ et $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\sin 2x} = \dfrac{5}{2}$.

Exercice 8

Calculer $\lim\limits_{x \to 1} \dfrac{\sin \pi x}{\sin 2\pi x}$.

Correction 8

$$\lim_{x \to 1} \frac{\sin \pi x}{\sin 2\pi x} = \lim_{h \to 0} \frac{\sin \pi(1+h)}{\sin 2\pi(1+h)} = \lim_{h \to 0} \frac{\sin(\pi + \pi h)}{\sin(2\pi + 2\pi h)} = \lim_{h \to 0} \frac{-\sin \pi h}{\sin 2\pi h} = \lim_{h \to 0} -\frac{\sin \pi h}{\sin 2\pi h}.$$

Or $\sin \pi h \underset{0}{\sim} \pi h$ et $\sin 2\pi h \underset{0}{\sim} 2\pi h$. Donc $\lim\limits_{x \to 1} \dfrac{\sin \pi x}{\sin 2\pi x} = -\dfrac{1}{2}$.

Exercice 9

Calculer $\lim\limits_{x \to 0} x \sin\left(\dfrac{1}{x}\right)$ et $\lim\limits_{x \to +\infty} x \sin\left(\dfrac{1}{x}\right)$.

Correction 9

- (Attention au cas $x < 0$) $0 \leq \left|x \sin\left(\dfrac{1}{x}\right)\right| \leq |x|$ donc $\lim\limits_{x \to 0} x \sin\left(\dfrac{1}{x}\right) = 0$.
- On pose $X = \dfrac{1}{x}$, $\lim\limits_{x \to +\infty} x \sin\left(\dfrac{1}{x}\right) = \lim\limits_{X \to 0} \dfrac{\sin X}{X} = 1$.

Exercice 10

Déterminer les limites suivantes :

1. $\lim\limits_{x \to +\infty} x \ln\left(1 - \dfrac{1}{x}\right)$.
2. $\lim\limits_{x \to +\infty} \sqrt{2x+5} \ln\left(1 + \dfrac{1}{\sqrt{2x+5}}\right)$.
3. $\lim\limits_{x \to +\infty} 3x^2 \ln\left(1 - \dfrac{1}{2x^2}\right)$.
4. $\lim\limits_{x \to +\infty} (2x+5) \ln\left(1 + \dfrac{1}{x-1}\right)$.
5. $\lim\limits_{x \to +\infty} x^2 \ln\left(1 - \dfrac{1}{x+3}\right)$.

Correction 10

- On pose $X = \dfrac{1}{x}$, $\lim\limits_{x \to \infty} x \ln\left(1 - \dfrac{1}{x}\right) = \lim\limits_{X \to 0} \dfrac{\ln(1-X)}{X} = -1$.
- On pose $X = \dfrac{1}{\sqrt{2x+5}}$, $\lim\limits_{x \to +\infty} \sqrt{2x+5} \ln\left(1 + \dfrac{1}{\sqrt{2x+5}}\right) = \lim\limits_{X \to 0} \dfrac{\ln(1+X)}{X} = 1$.
- On pose $X = \dfrac{1}{2x^2}$, on a $3x^2 = \dfrac{3}{2} \times 2x^2 = \dfrac{3}{2X}$.
  $\lim\limits_{x \to +\infty} 3x^2 \ln\left(1 - \dfrac{1}{2x^2}\right) = \lim\limits_{X \to 0} \dfrac{3}{2} \times \dfrac{\ln(1-X)}{X} = -\dfrac{3}{2}$
- $\lim\limits_{x \to +\infty} (2x+5) \ln\left(1 + \dfrac{1}{x-1}\right) = \lim\limits_{x \to +\infty} \dfrac{2x+5}{x-1} \times (x-1) \ln\left(1 + \dfrac{1}{x-1}\right)$.

$\lim\limits_{x \to +\infty} \dfrac{2x+5}{x-1} = 2$ et $\lim\limits_{x \to +\infty} (x-1) \ln\left(1 + \dfrac{1}{x-1}\right) = 1$

- $\lim\limits_{x \to +\infty} x^2 \ln\left(1 - \dfrac{1}{x+3}\right) = \lim\limits_{x \to +\infty} \dfrac{x^2}{x+3} \times (x+3)\ln\left(1 - \dfrac{1}{x+3}\right)$.

$\lim\limits_{x \to +\infty} \dfrac{x^2}{x+3} = +\infty$ et $\lim\limits_{x \to +\infty} (x+3)\ln\left(1 - \dfrac{1}{x+3}\right) = -1$.
Donc $\lim\limits_{x \to +\infty} x^2 \ln\left(1 - \dfrac{1}{x+3}\right) = -\infty$.

Exercice 11

Calculer $\lim\limits_{x \to +\infty} \left(\dfrac{x}{x+1}\right)^x$.

Correction 11

Rappel : Si $\lim\limits_{x \to a} f(x) = 1$ et $\lim\limits_{x \to a} g(x) = +\infty$ alors $\lim\limits_{x \to a} f(x)^{g(x)}$ est une forme indéterminée.
$\left(\dfrac{x}{x+1}\right)^x = \left(1 - \dfrac{1}{x+1}\right)^x = e^{x\ln\left(1-\frac{1}{x+1}\right)}$.
Or $\ln\left(1 - \dfrac{1}{x+1}\right) \underset{+\infty}{\sim} -\dfrac{1}{x+1}$ et $x\ln\left(1 - \dfrac{1}{x+1}\right) \underset{+\infty}{\sim} -\dfrac{x}{x+1}$.
Donc $\lim\limits_{x \to +\infty} x\ln\left(1 - \dfrac{1}{x+1}\right) = -1$ et $\lim\limits_{x \to +\infty} \left(\dfrac{x}{x+1}\right)^x = e^{-1}$.

Exercice 12

Calculer $\lim\limits_{x \to 0} (\cos x)^{1/x^2}$.

Correction 12

$(\cos x)^{1/x^2} = e^{\frac{1}{x^2}\ln\cos x}$. On a $\ln\cos x = \ln(1 + \cos x - 1)$. Or $\cos x - 1 \underset{0}{\sim} \dfrac{x^2}{2}$ et $\ln(1+x) \underset{0}{\sim} x$, donc $\ln\cos x \underset{0}{\sim} -\dfrac{x^2}{2}$ et $\dfrac{1}{x^2}\ln\cos x \underset{0}{\sim} -\dfrac{1}{2}$. D'où $\lim\limits_{x \to 0} (\cos x)^{1/x^2} = e^{-1/2}$.