On the internal dynamics of formations of unicycle robots

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Abstract—This paper studies a general class of formations of unicycle robots. One of the robots plays the role of the leader and the formation is induced through a constraint function $F$ that depends on the position and the orientation of the vehicles. We investigate the flexibility of such formations with respect to the leader’s reference frame by introducing the notion of formation internal dynamics, we characterize its equilibria and provide sufficient conditions for their existence. The theory is specialized to a particular constraint function $F$ that leads to a formation where robot $i$ follows a convex combination of the positions of the preceding $i-1$ robots. Sufficient conditions are presented in this scenario that guarantee that the position and orientation of the vehicles with respect to the leader’s reference frame are confined in a specific polyhedral region, regardless of the trajectory of the leader, provided that its curvature is sufficiently small.

I. INTRODUCTION

Over the past few years multi-agent systems and cooperative control have been the subject of vigorous research in the robotics and control communities [1], [2]. The reasons behind this success must be sought in multiple factors, such as, e.g., in the recent technological advances in wireless communications, in the wider availability of low-cost processing units, in the growing interest in parallel and embedded computing, in the definition of increasingly demanding tasks requiring the coordinate action of multiple autonomous agents to be accomplished.

Due to its wide applicability in real-world scenarios, the formation control problem has received special attention in the multi-agent systems literature. The idea behind formation control is that of controlling the relative position and orientation of the agents in a group, while allowing the group to move as a whole. Current work on formation control falls into three broad categories: virtual structure [3]–[5], behavior-based [6], and leader-follower [7]. This paper deals with a leader-following scheme.

Recently, the interest in formation control has been awakened by the introduction of the original notion of rigidity, that essentially measures how much a formation can be deformed by a smooth motion without affecting the relative distance between neighboring agents. Moving from [8], Anderson’s group has begun to systematically apply the rigid graph theory to the analysis of multi-agent formations and has shown the relevance of the rigidity notion in several engineering fields [9], [10]. In [11], [12], graph rigidity ideas have been used to design decentralized gradient control laws for the stabilization of a group of kinematic points to a target formation. Recently, in [13], a distributed algorithm that stabilizes the shape of a relative sensing network to a desired formation has been proposed: the algorithm relies on the global minimization of the stress majorization function associated to the network and represents an improvement over the approaches in [11], [12]. As it is evident from the previous literature review, most of the research in the rigidity framework has focused on graph-theoretical issues and little attention has been devoted to the physical constraints of the agents. In this respect, two challenging open problems include the characterization of rigidity for formations of nonholonomic mobile robots (that involve both distance and orientation constraints) and the definition of general classes of such formations sharing the same rigidity properties.

This paper draws partial inspiration from [14] and addresses these problems from a nonlinear control perspective. We first define a general class of formations of unicycle robots, where one of the robots acts as the leader and the formation is induced through a constraint function $F$ that depends on the position and the orientation of the vehicles. We then state conditions on $F$ that guarantee that the controls of the robots are uniquely defined. In this setting, the relative position and orientation of the followers with respect to the leader’s reference frame is not fixed (i.e. the formation is not rigid). The flexibility of the formation with respect to the leader’s reference frame is investigated by introducing the original notion of formation internal dynamics. The equilibrium configurations of the internal dynamics are characterized by two main results. A first theorem states that the relative distance and orientation of each follower with respect to the leader’s reference frame are fixed if and only if the vehicles either move along circular paths or parallel straight lines. A second theorem shows that such equilibria always exist if the trajectory of the leader is a circle of sufficiently small curvature or a straight line. The theory is illustrated on a specific constraint function $F$ that defines a formation where robot $i$ follows a convex combination of the positions of the preceding $i-1$ vehicles, thus generalizing previous results of the authors [15]. Some of the material of this work has already been announced in [16]: the main distinguishing feature of the present paper is a third theorem which provides sufficient conditions for the constraint function of Sect. III, that guarantee that the position and orientation of the robots relative to the leader are confined in a suitable polyhedral region, regardless of leader’s trajectory, as long as its curvature is sufficiently small.

The rest of the paper is organized as follows. Sect. II presents the theoretical foundation of the work. A case study is discussed in Sect. III and simulation results are reported in Sect. IV. In Sect V the main contributions of the article are summarized and future research directions are highlighted.

The following notation is used throughout the paper: $S^1$ denotes the quotient space $\mathbb{R}/2\pi\mathbb{Z}$, where $\mathbb{Z}$ is the set of integer numbers; $\forall x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ ($n \geq 1$), $(x, y) = \sum_{i=1}^{n} x_i y_i$. 

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Consider the following definition of robot as a velocity controlled unicycle model [15]:

**Definition 1 (Unicycle robot):** A function \( v = (x, y, \theta) \in C^1((0, +\infty), \mathbb{R}^2 \times S^1) \) is called a **unicycle robot** (or a **robot**, for short) if there exists a control function \( f(v, \omega) \in C^0((0, +\infty), \mathbb{R}^2) \) such that

\[
\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = \omega.
\]

For any \( t \in [0, +\infty) \), \((x(t), y(t))\) denotes the position of the robot at time \( t \), \( \theta(t) \) its heading, \( \tau(\theta(t)) \) the normalized velocity vector and \( \eta(\theta(t)) \) the normalized vector orthogonal to \( \tau(\theta(t)) \), (see Fig. 1). The next definition introduces an ordered set of \( n+1 \) robots, with \( n \geq 1 \).

**Definition 2 \((n+1)\)-tuple of robots:** Let

\[
\mathcal{X} = (\mathbb{R}^2 \times S^1)^{n+1} = \{ \xi = (\xi_0, \ldots, \xi_n) | \xi_i = (x_i, y_i, \theta_i) \in \mathbb{R}^2 \times S^1, \forall i = 0, \ldots, n \}.
\]

A \( C^1 \) map \( R : [0, +\infty) \rightarrow \mathcal{X} \) defined by \( R(t) = (r_0(t), \ldots, r_n(t)) \), where \( r_i = (x_i, y_i, \theta_i) \), \( i = 0, 1, \ldots, n \), is a robot, called an \( n+1 \)-tuple of robots. The set \( \mathcal{X} \) is called the configuration space, \( r_0 \) the leader and \( r_1, r_2, \ldots, r_n \) the followers.

Let us now introduce the following class of functions:

**Definition 3 (Constraint function):** Let \( F \) be the \( C^1 \) map defined on \( \mathcal{X} \) given by \( F = (F_1, F_2, \ldots, F_n) \), where \( F_i : \mathcal{X} \rightarrow \mathbb{R}^2, \forall i = 1, \ldots, n \). The map \( F \) is called a constraint function if \( \forall \xi \in \mathcal{X} \),

\[
\partial_{\xi_i} F_i(\xi) = 0, \quad i = 1, \ldots, n-1, \quad j = i+1, \ldots, n,
\]

i.e., every \( F_i \) depends only on \((\xi_0, \ldots, \xi_i)\).

**Definition 4 (Constraint set):** The set \( \mathcal{F} = \{ \xi \in \mathcal{X} | F(\xi) = 0 \} \) is called the **constraint set**: it is the set of configurations \( \xi \) compatible with the constraints \( F_1, F_2, \ldots, F_n \).

The constraint function \( F \) on the ordered set of \( n+1 \) robots, or \((n+1)\)-tuple of robots, imposes a structure on the set of constraints: in fact, constraint \( F_i \) depends only on the position and orientation of the robots that have index less or equal than \( i \). The following definition introduces the notion of **F-formation** used throughout the paper.

**Definition 5 (F-formation):** Let \( F \) be a constraint function. An \((n+1)\)-tuple of robots \( \mathcal{R} \) is said in **F-formation** if \( F(\mathcal{R}(t)) = 0, \forall t \geq 0 \), i.e., if \( \mathcal{R}(t) \in \mathcal{F}, \forall t \geq 0 \).

Set \( v_f = (v_1, v_2, \ldots, v_n) \) and \( \omega_f = (\omega_1, \omega_2, \ldots, \omega_n) \) where \((v_i, \omega_i)\) is the control function of \( i \)-th follower. We can then rewrite the kinematic equations of the \( n+1 \) robots in a compact form as,

\[
\dot{\xi} = g(\xi, v_0, \omega_0, v_f, \omega_f),
\]

where \((v_0, \omega_0)\) is the control of the leader and \( g = (g_0, g_1, \ldots, g_n) \) is such that \( \forall i = 0, \ldots, n: \)

\[
g_i(\xi, v_0, \omega_0, v_f, \omega_f) = (v_i \cos \theta_i, v_i \sin \theta_i, \omega_i).
\]

The following proposition presents a controlled invariance condition that plays a key role in the next developments.

**Proposition 1:** Let \( F \) be a constraint function and suppose that \( \forall i = 1, \ldots, n, \forall \xi \in \mathcal{F}: \)

\[
\det(\cos \theta_i \partial_{x_i} F_i(\xi) + \sin \theta_i \partial_{y_i} F_i(\xi), \partial_{\theta_i} F_i(\xi)) \neq 0.
\]

Then the following facts hold:

i) \( F \) is a differential manifold of dimension \( n+3 \).

ii) For each \( \xi \in \mathcal{F} \) and for each \((v_0, \omega_0)\) there exist unique controls \( v_f(\xi, v_0, \omega_0), \omega_f(\xi, v_0, \omega_0) \) such that \( g(\xi, v_0, \omega_0, v_f(\xi, v_0, \omega_0), \omega_f(\xi, v_0, \omega_0)) \in T_\xi \mathcal{F} \).

**Proof:** Denote by \( F(\xi) \) the Jacobian matrix of \( F \). Since \( \text{rank}(F(\xi)) = 2n \) by (3), we get that \( i \) holds by the implicit function theorem. To prove ii), set \( \forall i = 1, \ldots, n, \forall j = 0, \ldots, n, \partial_{x_j} F_i(\xi) = (\partial_{x_j} F_i(\xi), \partial_{y_j} F_i(\xi), \partial_{\theta_j} F_i(\xi)) \), which is a \( 2 \times 3 \) matrix, and \( \forall i = 1, \ldots, n, A_{ij}(\xi) = \partial_{\xi_j} F_i(\xi) = (\cos \theta_i \sin \theta_i, \sin \theta_i \sin \theta_i) \).

Then, by hypothesis (3), \( \forall (v_0, \omega_0) \in \mathbb{R}^2, v_f = (v_1, v_2, \ldots, v_n) \), \( \omega_f = (\omega_1, \omega_2, \ldots, \omega_n) \) are given by the unique solution of the triangular system \( A_{ij}(\xi) (v_i, \omega_i)^T = -\sum_{j=0}^{i-1} A_{ij}(\xi) (v_j, \omega_j)^T, i = 1, \ldots, n. \)

**Remark 1:** Condition (3) in Proposition 1 guarantees that if the vehicles are in \( F \)-formation at time \( t = 0 \), then there exist unique controls \( v_f, \omega_f \) for the followers such that they remain in \( F \)-formation for all \( t \geq 0 \), independently of leader’s controls \( v_0, \omega_0 \).

The following proposition shows that a regularity condition slightly stronger than (3) guarantees the **local stabilizability** of the formation, i.e., there exists a neighborhood of \( \mathcal{F} \) such that if \( R(0) \) belongs to this set, we can find controls for the followers that allow them to be asymptotically in formation, regardless of leader’s trajectory.

**Proposition 2 (Local stabilizability):**

Let \( \mathcal{F}(\varepsilon) = \{ \xi \in \mathcal{X} | ||F(\xi)|| \leq \varepsilon \} \). Suppose that there exists \( \varepsilon > 0 \) such that, \( \forall \xi \in \mathcal{F}(\varepsilon), \)

\[
\det(\cos \theta_i \partial_{x_i} F_i(\xi) + \sin \theta_i \partial_{y_i} F_i(\xi), \partial_{\theta_i} F_i(\xi)) \neq 0.
\]

Then \( \forall (v_0, \omega_0) \in C^0([0, +\infty), \mathbb{R}), \) there exist control functions \( v_f(\xi, v_0, \omega_0), \omega_f(\xi, v_0, \omega_0) \in C^0([0, +\infty), \mathbb{R}) \) such that \( \forall \xi \in \mathcal{F}(\varepsilon), \) the solution \( R(t) \) of

\[
\dot{\xi} = g(\xi, v_0, \omega_0, v_f(\xi, v_0, \omega_0), \omega_f(\xi, v_0, \omega_0)), \xi(0) = \xi, \quad \text{is such that } \lim_{t \to +\infty} ||F(R(t))|| = 0.
\]

**Proof:** By hypothesis (4), using the notation of the proof of Proposition 1, it suffices to define \( v_f, \omega_f \) in such a way that \( A_{ij}(\xi) = \sum_{j=0}^{i-1} A_{ij}(\xi) (v_j, \omega_j)^T - k F_i(\xi), \forall i = 1, \ldots, n, k > 0, \) is solved. Therefore \( \forall \xi \in \mathcal{F}(\varepsilon), \forall t \geq 0, \forall i = 1, \ldots, n, \) the solution \( R(t) \) of system (5) has the property \( \frac{1}{k} F_i(R(t)) = -k F_i(R(t)), \forall t \geq 0, \forall i = 1, \ldots, n, \) which implies the thesis.
**Definition 6 (Regular constraint function):** Let $F$ be a constraint function. $F$ is called regular if condition (3) is satisfied.

**Remark 2:** In Definition 3 we introduced $2n$ scalar constraints on the $(n+1)$-tuple of robots. This is a necessary condition for the uniqueness of the controls $v_f, \omega_f$ and it is not a loss of generality since with less constraints it is always possible to restate the problem by adding a suitable set of fictitious ones.

To properly study the relative interaction between the robots, we now introduce the notion of *rototranslation invariance*, i.e., a function invariant to a rigid body motions and changes of orientation.

**Definition 7 (Rototranslation invariance):** For any parameter $p = (x, y, \theta) \in \mathbb{R}^2 \times S^1$, define the map $H_p : \mathbb{R}^2 \times S^1 \rightarrow \mathbb{R}^2 \times S^1$ as $H_p(x, y, \theta) = ((x + \bar{x}, y + \bar{y})^T, \tau + \theta)$ where $\tau(\theta) = (\tau(\theta), \eta(\theta))$. For any $\xi = (\xi_0, \ldots, \xi_n) \in \mathcal{X}$, set $H_p(\xi) = (H_p(\xi_0), \ldots, H_p(\xi_n))$. A map $G \in \mathcal{C}(\mathcal{X}, \mathbb{R}^n)$ is called rototranslation invariant if $G(\xi) = G(H_p(\xi))$, $\forall \xi \in \mathcal{X}, \forall p \in \mathbb{R}^2 \times S^1$.

**Remark 3:** Let $F$ be a rototranslation invariant constraint function and suppose that the $(n+1)$-tuple of robots $\mathcal{R} = (r_0, r_1, \ldots, r_n)$ is in $F$-formation. Therefore for any $\lambda \in \mathcal{C}((0, +\infty), \mathcal{R})$, $\mathcal{R}(t) = \mathcal{R} \circ \lambda$ is in $F$-formation. Whence, if $|v_0(t)| \neq 0$, $\forall t \geq 0$, taking $\lambda(t) = \int_0^t v_0^1(\tau) d\tau$, it follows that the forward velocity of the leader of $\mathcal{R}$ is always 1.

This justifies the following assumption:

**Assumption 1:** We henceforth suppose that the forward velocity $v_0$ of the leader $r_0$ is equal to 1.

As a consequence, $\omega_0(t)$ represents the *curvature* of the path $(x_0(t), v_0(t))$ followed by $r_0$ at time $t$. Let us now introduce the following *equivalence relation* $\sim$ on $\mathcal{X}$.

**Definition 8:** Given $\xi, \vartheta \in \mathcal{X}$, $\vartheta \sim \xi$ if there exists a $p \in \mathbb{R}^2 \times S^1$, such that $\vartheta = H_p(\xi)$.

Let $\mathcal{E} = \{\vartheta \in \mathcal{X} / \vartheta \sim \xi\}$ and denote by $\mathcal{X} / \sim \rightarrow \mathcal{E}$ the quotient set. Note that $G$ is a rototranslation invariant function if and only if $\forall \xi \in \mathcal{E}, G$ is constant on $[\xi]$. The set of rototranslation invariant constraints functions, it is natural to define a reduced constraint set as follows:

**Definition 9 (Reduced constraints set):** The reduced constraints set is the set $\mathcal{F}_r = \mathcal{F} / \sim$.

Owing to this definition, each element of $\mathcal{F}_r$ represents a set of configurations for the formation that differ only by a rototranslation. The proof of the next proposition is given in [16, Prop. 2].

**Proposition 3:** Let $F$ be a regular rototranslation invariant constraint function and $\mathcal{R}^1, \mathcal{R}^2$ be two $(n+1)$-tuples of robots in $F$-formation. Suppose that, $\mathcal{R}^1(0) \sim \mathcal{R}^2(0)$, then $\mathcal{R}^1(t) \sim \mathcal{R}^2(t), \forall t \geq 0$.

Let $\omega_0 \in \mathcal{C}((0, +\infty), \mathcal{R})$ be the control for the leader. Owing to Proposition 3 the following map is well defined:

$$\Phi_{\omega_0} : [0, +\infty) \times \mathcal{F}_r \rightarrow \mathcal{F}_r, (t, [\xi]) \mapsto \Phi_{\omega_0}(t, [\xi]),$$

$$\Phi_{\omega_0}(t, [\xi]) = [\xi(t)], \quad \text{(6)}$$

being $\xi$ the only solution of the system $\dot{\xi} = g(\xi, t, \omega_0, \gamma_f(\xi, t, \omega_0), \gamma_f(\xi, 0, \omega_0))$, $\xi(0) = \xi$ where $g(\cdot)$ is given by (2).

**Remark 4:** Let $F$ be a rototranslation invariant constraint function. Set $\Gamma = \{\xi = (\xi_0, \ldots, \xi_n) \in \mathcal{X} \mid \xi_0 = 0, F(\xi) = 0\}$. Then the map, $F_r \rightarrow \Gamma, [\xi] \mapsto H_{-\xi_0}(\xi)$, is a bijection. Moreover an $(n+1)$-tuple of robots $\mathcal{R}$ is in $F$-formation if and only if $H_{-\xi_0}(\mathcal{R}(t)) \in \Gamma, \forall t \geq 0$.

As suggested by (7), $\mathcal{F}_r$ is the set of all configurations of followers in the leader’s reference frame that are compatible with the constraint function.

**Definition 10 (Reduced motion):** Given an $(n+1)$-tuple of robots $\mathcal{R}$, the reduced motion of $\mathcal{R}$ is the map $[\mathcal{R}] : [0, +\infty) \rightarrow \mathcal{X} / \sim$ defined by $[\mathcal{R}](t) = [\mathcal{R}(t)], \forall t \geq 0$.

Note that $[\mathcal{R}]$ describes the motion of the followers in the leader’s reference frame.

**Remark 5:** Let $\mathcal{R} = (r_0, \ldots, r_n)$ be an $(n+1)$-tuple of robots. $[\mathcal{R}]$ is constant if and only if there exists $\xi \in \Gamma$ such that $H_{-\xi_0}([\mathcal{R}](t)) = \xi, \forall t \geq 0$.

Since $\mathcal{F}_r$ is the quotient set of $\mathcal{F}$ by the equivalence relation $\sim$, it has a lower dimension than $\mathcal{F}$ as specified in the following proposition (see [16, Prop. 3] for the proof).

**Proposition 4:** If $F$ is a regular rototranslation invariant constraint function, $\mathcal{F}_r$ is a differential manifold of dimension $n$, diffeomorphic to $\Gamma$.

**Remark 6 (Flexibility of $F$-formations):** A consequence of Proposition 4 is that the $F$-formations are not rigid. This is because we are dealing with robots described by 3 configuration variables, but $F$ introduces only 2 constraints for each robot: this results in a residual degree of freedom for each vehicle. More precisely, we can say that a formation is rigid if the set $\mathcal{F}_r$ is composed by only one equivalence class; this is not true for $F$-formations since $\mathcal{F}_r$ is a manifold of dimension $n$.

**Definition 11 (Formation internal dynamics):** Suppose $F$ is a regular rototranslation invariant constraint function such that, $\forall \xi \in \Gamma$,

$$\det(\partial_{x_1}, F(\xi), \partial_{y_1}, F(\xi), \ldots, \partial_{x_n}, F(\xi), \partial_{y_n}, F(\xi)) \neq 0.$$  

(8)

By the implicit function theorem, there exists a $C^{1}$ diffeomorphism $\gamma : T^n \rightarrow \Gamma, \beta = (\beta_1, \ldots, \beta_n) \sim \gamma(\beta)$ where $T^n$ is the $n$-torus. Let us identify $\mathcal{F}_r$ with $\Gamma$ and $\forall \xi \in \Gamma, \forall \omega_0 \in C^0([0, +\infty), \mathcal{R})$, let $\Phi_{\omega_0} : [0, +\infty) \times \Gamma \rightarrow \Gamma, (t, \xi) \mapsto \Phi_{\omega_0}(t, \xi)$, be the map given by (6). Then for any $\beta \in T^n$, set $\beta(t, \beta, \omega_0) = \gamma^{-1}(\Phi_{\omega_0}(t, \gamma(\beta))), \forall t \geq 0$, which belongs to $C^1([0, +\infty), T^n)$. Then $\beta(t, \beta, \omega_0)$ is the unique solution of the following system,

$$\dot{\beta}(t) = h(\beta(t), \omega_0(t)), \quad \beta(0) = \bar{\beta},$$

(9)

where $\forall t \in [0, +\infty), \forall \beta \in T^n : h(\beta, \omega_0) = \frac{\partial}{\partial \beta}(\gamma^{-1}(\Phi_{\omega_0}(t, \gamma(\beta))))(0)$. System (9) is called *reduced internal dynamics* of the formation.

System (9) describes the motion of the followers in the reference frame of the leader. Note that the equilibria of the function $h$ represent the configurations that are constant in the leader’s reference frame.

The following theorem states that if property (10) (see the statement below) holds, then the distance and relative orientation of each follower with respect to the leader’s reference frame are fixed if and only if the robots either move along circular paths or parallel straight lines. The proof of the theorem is omitted due to space limitations.
Theorem 1: Suppose that $F$ is a regular rototranslation invariant constraint function and that:

\[
\mathcal{F} \text{ does not contain any point of the set:} \\
\{ \xi \in \mathcal{X} \mid ((x_i - x_0, y_i - y_0)^T, \tau(\theta_0)) = 0, \ theta_i \in \{\theta_0, \theta_0 + \pi\}, \forall i = 1, \ldots, n\}.
\]

Suppose that an $(n+1)$-tuple of robots $\mathcal{R}$ is in $F$-formation. Then the following properties are equivalent:

(i) $[\mathcal{R}]$ is constant.

(ii) $\omega_0$ is constant, moreover:

a) if $\omega_0 = 0$, then $\theta_0(t) = \bar{\theta}_0$, $\forall t \geq 0$ and $\forall i = 1, \ldots, n$, $\theta_i(t) \in \{\theta_0, \theta_0 + \pi\}$, $\forall t \geq 0$.

b) if $\omega_0 \neq 0$, then there exists $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ such that $\forall i = 0, \ldots, n$, $((x_i(t), y_i(t))^T - (\bar{x}, \bar{y})^T, \tau(\theta_i(t))) = 0$, $\forall t \geq 0$.

From a geometric viewpoint, the “nonalignment condition” (10) means that the manifold $\mathcal{F}$ must not contain configurations in which all the followers are placed on a straight line passing through the leader and orthogonal to $\tau(\theta_0)$, with the robots all oriented in the same direction.

Definition 12 (Nice constraint function): Let $F$ be a constraint function. $F$ is called a nice constraint function if it satisfies the following four properties: $i)$ $F$ is regular, $ii)$ $F$ is rototranslation invariant, $iii)$ $F$ verifies property (10), $iv)$ $F$ verifies property (8).

The following theorem provides a sufficient condition for the existence of the equilibrium configurations introduced in Theorem 1, when the angular velocity $\omega_0$ of the leader is constant and sufficiently small: for the sake of brevity, the proof is omitted (see [16, Th. 2]). For any $i = 1, \ldots, n$, let $\Pi_i : \mathcal{X} \rightarrow \mathbb{R}^2$ be the linear function $\Pi_i(\xi) = (x_i, y_i).

Theorem 2: Let $F$ be a nice constraint function. Set,

\[
\bar{\rho} = \sup \{||\Pi_i(\xi)|| \mid \forall \xi \in \Gamma, \forall i = 1, \ldots, n\}.
\]

Then $\bar{\rho} > 0$ and for any $\bar{\omega}$ such that

\[
-1/\bar{\rho} < \bar{\omega} < 1/\bar{\rho},
\]

there exists an $(n + 1)$-tuple of robots $\mathcal{R}$ such that $[\mathcal{R}]$ is constant and $\omega_0(t) = \bar{\omega}$, $\forall t \geq 0$.

III. AN APPLICATION TO A SPECIFIC CONSTRAINT FUNCTION OF HIERARCHICAL TYPE

In this section the theoretical results presented in Sect. II are illustrated on a specific constraint function. This particular function may be useful to describe formations occurring in nature, such as, e.g., bird flocks, where is believed that each animal follows an average of the preceding birds [17]. To define $F = (F_1, \ldots, F_n)$, assign the following constant parameters $\forall i = 1, \ldots, n$, $\lambda_{i,0}, \ldots, \lambda_{i-1} \geq 0$, $\sum_{k=0}^{i-1} \lambda_{i,k} = 1$, $d_i > 0$, $|\phi_i| < \pi/2$, and set

\[
F_i(x_0, x_1, \ldots, x_i) = \sum_{k=0}^{i-1} \lambda_{i,k} (x_k - x_i) - d_i \tau(\theta_i + \phi_i).
\]

With this constraint function, we require robot $i$ to follow a convex combination of the positions of the preceding $i-1$ robots at a fixed distance $d_i$ and with a fixed visual angle $\phi_i$ (see Fig. 2). Note that, if $\forall i = 1, \ldots, n$, there exists an integer $l_i$ such that $0 \leq l_i \leq i - 1$ and $\lambda_{i,k} = \delta_{k,l_i}$, where $\delta_{k,l}$ denotes the Kronecker’s delta, then $F$ induces the kind of hierarchical formations studied in [15]. $F$ is a regular constraint function since (1) holds and (3) is satisfied. In fact $\det(\cos \theta_i \partial_x F_i + \sin \theta_i \partial_y F_i) = d_i \cos \phi_i$ and $|\phi_i| < \pi/2$ by hypothesis. Clearly $F$ is rototranslation invariant and it verifies (10) and (8). Therefore $F$ is a nice constraint function and we can apply the theory developed in Sect. II. In particular, from Proposition 1 we have that if $\mathcal{R}$ is an $(n+1)$-tuple of robots which is in $F$-formation at the initial time (i.e., $F(\mathcal{R}(0)) = 0$), then for any trajectory of the leader (i.e., $\forall \omega_0 \in C^0([0, +\infty[; \mathcal{X})$, recall Assumption 1), there exist and are unique the controls $v_f = (v_1, \ldots, v_n)$, $\omega_f = (\omega_1, \ldots, \omega_n)$ for the followers, such that $\mathcal{R}$ is in $F$-formation for any $t \geq 0$ (i.e., $F(\mathcal{R}(t)) = 0$, $\forall t \geq 0$). These controls are given by

\[
v_i(t) = \frac{1}{\cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k(t) \cos(\theta_k(t) - \theta_i(t) - \phi_i),
\]

\[
\omega_i(t) = \frac{1}{d_i \cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k(t) \sin(\theta_k(t) - \theta_i(t)).
\]

From Theorem 1 we know that an $(n+1)$-tuple of robots $\mathcal{R}$ in $F$-formation has a fixed configuration in the leader’s reference frame (i.e., $[\mathcal{R}(t)]$ is constant) if and only if the robots are moving along straight lines or circles. From Theorem 2 we have that such a fixed relative configuration exists when $\omega_0$ is constant and $|\omega_0| < 1/\bar{\rho}$ where $\bar{\rho}$ is given by (11). In our specific case $\bar{\rho}$ can be bounded from above as follows,

\[
\bar{\rho} \leq \max_{i=1,\ldots,n} \bar{\rho}_i,
\]

where $\bar{\rho}_i = d_i + \sum_{k=0}^{i-1} \lambda_{i,k} \bar{\rho}_k$ and $\bar{\rho}_0 = 0$. Moreover, the formation internal dynamics is given by

\[
\dot{\beta}_i(t) = h_i(\beta(t), \omega_0(t)),
\]

where $\forall \beta = (\beta_1, \ldots, \beta_n) \in T^n$,

\[
h_i(\beta, \omega_0) = -\omega_0 - \frac{1}{d_i \cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k(\beta) \sin(\beta_i - \beta_k),
\]

\[
v_i(\beta) = \frac{1}{\cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k(\beta) \cos(\beta_k - \beta_i - \phi_i),
\]

and $\beta_0 = 0$. System (16) has been obtained by computing the time derivative of $\beta_i = \theta_i - \theta_0$ and using (14).
Given an \((n + 1)\)-tuple of robots in \(F\)-formation, the following theorem provides a method to determine a polyhedral invariant region for the formation internal dynamics (16), whose size depends on the bounds on leader’s curvature \(\omega_0\).

**Theorem 3 (Invariant region for (16)):** Let \(v_0 = 1\) and let \(\omega_0^+\) be real constants. Set \(V = \{1\} \times \{v_1^-, v_1^+\} \times \cdots \times \{v_{n-1}^-, v_{n-1}^+\}\) and define the following set of constants:

\[
\beta_0^+ = 0 \quad \text{and} \quad \beta_i^+ = \max\{\text{arg}(\sum_{k=0}^{i-1} \lambda_{i,k} v_k e^{i\beta_k}) - \arcsin\left(\frac{\omega_0}{|\sum_{k=0}^{i-1} \lambda_{i,k} v_k e^{i\beta_k}|}\right)\}, \quad \beta_i^- = \min\{\text{arg}(\sum_{k=0}^{i-1} \lambda_{i,k} v_k e^{i\beta_k}) - \arcsin\left(\frac{\omega_0}{|\sum_{k=0}^{i-1} \lambda_{i,k} v_k e^{i\beta_k}|}\right)\}
\]

where

\[
\lambda_{i,k} = \sqrt{-1}, \quad v_0^- = v_0^+ = 1, \quad v_i^- = \frac{\cos \phi_i}{\sum_{k=0}^{i-1} \lambda_{i,k} v_k e^{i\beta_k}} \cos (\max\{0, -\beta_k\}) - \phi_i - (\beta_k - \beta_i^-), \quad v_i^+ = \frac{\cos \phi_i}{\sum_{k=0}^{i-1} \lambda_{i,k} v_k e^{i\beta_k}} \cos (\max\{0, -\beta_k\}) - \phi_i - (\beta_k - \beta_i^+),
\]

\(\phi_i = \min\{0, -\beta_k\}\) if \(a_{i,k} = 0\) and \(a_{i,k} = 0\) otherwise. Set \(B = [\beta_1^+, \beta_1^-] \times \cdots \times [\beta_n^+, \beta_n^-]\). Suppose that

\[
\beta_i^+ \geq 0, \quad |\beta_i^-| < \pi/4, \quad \text{and that} \quad \omega_0 \in C^0([0, +\infty], \mathbb{R}) \text{ is such that,}
\]

\[
\beta_i^+ \leq 0 \quad \text{and} \quad \beta_i^- \leq v_i(t) \leq \beta_i^+ \quad \forall t \geq 0.
\]

Before proceeding to the proof of Theorem 3, we make the following remarks:

a) Property (22) guarantees that while the vehicles move, the relative orientation between robot \(i\) and the leader is bounded by the constants \(\beta_i^\pm\). This condition limits the possible variations in shape of the formation, in the leader’s reference frame.

b) There always exist (sufficiently small) values of positive and negative curvature for the leader such that the hypotheses of Theorem 3 are satisfied and condition (22) holds. In fact, when \(\omega_0^- = 0\), then, for all \(i = 1, \ldots, n\), \(v_i^- = v_i^+ = 1\) and hence condition (20) of Theorem 3 is verified. This implies, by continuity, that there exists a real constant \(\epsilon > 0\) such that if \(\omega_0^- < \epsilon\), then (20) is still met.

c) Once the weights \(\lambda_{i,k}\) of function (13) have been assigned, one can compute \(v_i^-, v_i^+, \beta_i^-, \beta_i^+\) for all values of \(\omega_0^-\) and \(\omega_0^+\) for which (20) is satisfied. In this way it is possible to obtain bounds on the velocity of the followers and on their misalignment with respect to \(r_0\) for different choices of the maximum curvature allowed to the leader.

**Proof of Theorem 3** Let \(\omega_0 \in C^0([0, +\infty], \mathbb{R})\) be such that (21) holds and let \(\beta \in C^1([0, +\infty], T^n)\) be a solution of system (16) such that \(\beta(0) \in B\). Note that (22) and (23) hold if we show that the following property is true (recall Assumption 1):

\[
\begin{align*}
\beta_i^+ &\leq \beta_k(t) \leq \beta_i^- \quad \forall t \geq 0, \\
\beta_i^- &\leq \beta_i(t) \leq \beta_i^+ \quad \forall t \geq 0,
\end{align*}
\]

By (24), we observe that \((v_1(t), \ldots, v_{n-1}(t)) \in [v_1^-, v_1^+] \times \cdots \times [v_{n-1}^-, v_{n-1}^+]\). Note that \(V\) represents the set of vertices of the polyhedron \(\{1\} \times \{v_1^-, v_1^+\} \times \cdots \times \{v_{n-1}^-, v_{n-1}^+\}\) and that the cardinality of this set is \(2^{n-1}\) (we shall denote a generic element of \(V\) by \(z = (z_0, \ldots, z_{n-1})\)). Then, if (24) is verified, there exist \(2^{n-1}\) nonnegative continuous functions \(\mu_k(t) \in V\) such that \(\sum_{k \in V} \mu_k = 1 \quad \text{and} \quad (1, v_1(t), \ldots, v_{n-1}(t)) = \sum_{k \in V} \mu_k(t) z_k\).

Therefore,

\[
h_i(\beta(t), \omega_0(t)) = \omega_0(t) - \sum_{k=0}^{i-1} \lambda_{i,k} \sum_{\mu_k \in V} \mu_k(t) z_k \times
\]

\[
\sin(\beta_i(t) - \beta_i(t)) = \sum_{\mu_k \in V} \mu_k(t) h_i(\beta(t), \omega_0(t)),
\]

(27)

where \((\sum_{\mu_k \in V} \mu_k(t) z_k)_{k=0}^{i-1}\) denotes the \(k\)-th component of \(\sum_{\mu_k \in V} \mu_k(t) z\) and \(\forall z \in V, h_i(\beta, \omega_0) = \omega_0 - \sum_{k=0}^{i-1} \lambda_{i,k} z_k \sin(\beta_i - \beta_i)\). Set \(S_+ = [\beta_1^+, \beta_1^-] \times \cdots \times [\beta_{n-1}^+, \beta_{n-1}^-] \times \{\beta_n^+\}, \quad S_- = [\beta_1^-, \beta_1^+] \times \cdots \times [\beta_{n-1}^-, \beta_{n-1}^+] \times \{\beta_n^-\}\), then by (19) it follows that \(\forall z \in V, h_i(\beta, \omega_0) \leq 0, \quad \forall \beta \in S_+, \quad \forall \omega_0 \in [\omega_0^-, \omega_0^+], \quad \text{and} \quad h_i(\beta, \omega_0) \geq 0, \quad \forall \beta \in S_-\), \(\forall \omega_0 \in [\omega_0^-, \omega_0^+], \) which, implies by (27), that \(h_i(\beta, \omega_0) \leq 0, \quad \forall \beta \in S_+, \quad \text{and} \quad h_i(\beta, \omega_0) \geq 0, \quad \forall \beta \in S_-\), and that (25) holds. Finally, inequality (26) is true by direct computation.

**IV. SIMULATION RESULTS**

Numerical simulations have been performed using the constraint function (13) with \(d_1 = d_2 = d_3 = 1\) and \(\phi_1 = \phi_2 = \phi_3 = \pi/4\). We set \(\lambda_{i,k} = 1/i, \quad i = 1, 2, 3, \quad k = 0, \ldots, i - 1\); in this way robot \(i\) follows exactly the average of the positions of the preceding \(i - 1\) robots in the formation. Three different trajectories have been assigned to the leader \(r_0\). In the first simulation the leader moves along a circular path with constant curvature \(\omega_0(t) = 0.25\) (a circle of radius 4). At the initial time \(t = 0\) the robots are in formation and all aligned with the \(x\)-axis (i.e., \(\theta_i(0) = 0, \quad i = 0, 1, 2, 3\)). The formation asymptotically reaches an equilibrium configuration in which all the robots move along concentric circles (see Fig. 3(a)) and each follower occupies a fixed position in the reference frame of the leader. This equilibrium configuration exists by virtue of Theorem 2, because in this case, using (15), we obtain \(1/\beta = 0.546\). Since condition (20) is satisfied, we can apply Theorem 3. The time history of the internal dynamics variables \(\beta_i\) is reported in Fig. 3(d) where the dashed lines represent the bounds \(\beta_i^\pm\). Note that in this case the bounds are very tight: all the variables \(\beta_i\) asymptotically converges to a constant value. In the second simulation, the leader moves along a circle of radius 1 (hence, \(\omega_0 = 1\)). In this case hypothesis (12) of Theorem 2 is not satisfied. Furthermore, we cannot apply Theorem 3 since condition (20) is not verified. In fact, the followers do not converge to circular trajectories (see Fig. 3(b)) and the variables \(\beta_2\) and \(\beta_3\) diverge (see Fig. 3(e)). Finally, in the third simulation the leader has a time-varying curvature: \(\omega_0(t) = 0.25 \sin(t/4)\). The trajectories of the robots are depicted in Fig. 3(c). As in
the first simulation condition (20) is satisfied and Theorem 3 can be applied. The time history of $\beta_i$ and the corresponding bounds $\beta_i^{\pm}$ (dashed lines) is reported in Fig. 3(f).

V. CONCLUSIONS AND FUTURE WORK

In this paper we have introduced a general class of formations of unicycle robots. One of the robots acts as the leader and the formation is induced through a constraint function $F$ that depends on the pose of the vehicles. We have studied the flexibility of such formations with respect to the leader’s reference frame by introducing the notion formation internal dynamics, characterized its equilibria, and given sufficient conditions for their existence (Theorems 1 and 2). The theory is illustrated on a constraint function of special structure: a polyhedral invariant region has been analytically determined in this case for the formation internal dynamics (Theorem 3). Work is in progress to extend the validity of our results to broader classes of formations of nonholonomic robots and to explore alternative definitions of formation internal dynamics. We also aim at studying the flexibility of the architectures considered in [7, Sects. VIA-B], and at exploiting the tools developed in this paper to quantify the error amplification in large leader-follower formations.

REFERENCES