Abstract— This paper studies the active target-tracking problem for a team of unmanned aerial vehicles equipped with 3-D range-finding sensors. We propose a gradient-based control strategy that encompasses the three major optimum experimental design criteria, and we use the Kalman filter for estimating the target’s position both in a cooperative and in a noncooperative scenario. Our control strategy is active because it moves the vehicles along paths that minimize the uncertainty about the location of the target. In the case that the position of the vehicles is not perfectly known, we introduce a new and more challenging problem, termed “Active Cooperative Localization and Multi-target Tracking” (ACLMT). In this problem, the aerial vehicles must reconfigure themselves in the 3-D space in order to maximize both the accuracy of their own position estimate and that of multiple moving targets. For ACLMT, we derive analytical lower and upper bounds on the targets’ and vehicles’ position uncertainty by exploiting the monotonicity property of the Riccati differential equation arising from the Kalman-Bucy filter. These bounds allow us to study the impact of sensors’ accuracy and targets’ dynamics on the performance of our coordination strategy. Extensive simulation experiments illustrate the proposed theoretical results.

Index Terms— Active sensing, mobile sensors, unmanned aerial vehicles, target tracking, cooperative localization, Kalman filtering

I. INTRODUCTION

A. Problem statement and related work

This paper deals with the optimal deployment of a team of unmanned aerial vehicles in the 3-D space, in order to maximize the accuracy of the position estimate of one or multiple moving targets (active target-tracking problem). This problem has relevant applications in surveillance, patrolling, military and environmental monitoring tasks, and offers different levels of complexity depending on whether the position of the agents is perfectly known or needs to be estimated together with that of the targets (localization problem), and on whether a collaborative or a noncooperative approach is adopted. Since testing on real flying platforms is complex, expensive and time-consuming, analytical tools are needed to predict the performance of a newly-synthesized algorithm in an early design stage. This paper moves towards this direction and presents analytical results elucidating the role played by the targets’ dynamics and the accuracy of the on-board range-finding sensors, on the (steady-state) target-tracking and vehicle localization performances.

Multi-agent systems research has lately gained a prominent role in the robotics and control literature [1]–[3]. The surge of interest in this subject has been mainly driven by the sheer increase in autonomy, computing power and sensing of today’s robots. An essential capability that each autonomous robot should possess is that of being able to efficiently measure the surrounding environment and to promptly respond to stimuli coming from other robots, humans, or moving targets. Extensive research has been done in the literature on the subject of tracking targets with static sensors [4], [5], or on the optimal placement of fixed sensors [6]. Mobile sensor networks are known to offer distinctive advantages over static sensor arrays in terms of quality of sensing and estimation, area coverage, adaptability to changing conditions (in the environment as well as in the target’s behavior), and robustness against failures. Cooperative active sensing leverages the mobility of a robotic sensor network in order to enhance the target tracking performances [7]–[18]: in the simplest instance of this problem, n sensors have to fuse their local measurements and move in order to attain the best position estimate of a moving target. This mechanism is sometimes referred to as “information-driven mobility” in the literature, and it amounts, in practice, to minimize a certain scalar function of the covariance matrix of the position estimates: in optimum experimental design theory [6], [19], this function is typically the determinant or the trace of the covariance matrix (D- and A-optimality criteria).

In the cooperative active target-tracking literature, a large body of research has focused on agents equipped with sensors providing range-only or range-bearing measurements to the target, while relatively fewer works have dealt with the more challenging bearing-only case (c.f. [20]). Moreover, most of the existing papers have limited themselves to single targets moving in a 2-D environment, and none of them has conducted an analytical study of the impact of system’s parameters on the performance of the proposed coordination strategies.

In [7], a motion-planning algorithm has been presented for solving the cooperative 2-D target-tracking problem using range-bearing measurements. The control law proposed by the authors is based on the gradient of the determinant of the covariance matrix of the target’s position estimate with respect to each of the robot’s coordinates. The multiple-target case is briefly treated as a possible extension, and the role of imperfect communication between agents is also investigated, thus elucidating the trade-offs in performance between sensing and communication. The results in [7] have been extended in two directions in [8]. First, by using dynamic average consensus estimators, the controller proposed in [7] is made

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Active Target Tracking and Cooperative Localization for Teams of Aerial Vehicles

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distributedly implementable. Second, a new control design procedure based on the distributed Kalman filter [21] for estimation fusion is presented, and the case of range-only sensors is addressed. Pursuing this line of research, the same authors have lately proposed decentralized estimation algorithms alternative to the Kalman filter (\(H_\infty\) filters, set-valued estimators), and studied the impact of “intelligent” evasive targets on the tracking performances. Embracing an approach similar to [8], the collaborative active-sensing problem has been studied in [9] for a group of double-integrator ground robots equipped with range-bearing sensors. The network-connectivity issue is addressed using a flocking-based mobility model (which also accounts for collision avoidance), and a modified version of the distributed Kalman filter in [21] for estimating the target’s state is presented, along with information-driven decentralized control laws for the agents. Recently, in [10], a formal closed-loop stability analysis of an enhanced version of the coupled estimation-and-control algorithm described in [9], has been proposed by the same author. However, both in [9] and [10], only the single-target case is addressed. In [11], the authors focused on discrete-time target-tracking for a team of unicycle robots with bounds on the positive forward velocity, and shown that the associated optimization problem is NP-hard in general. Non-trivial relaxations to this problem are then proposed for determining the set of feasible locations that each robot should move to, in order to collect the most informative distance measurements. Note that differently from [7], where the nonlinearities inherent to the range-bearing model are “absorbed” into the nonstationary sensor noise, the approach in [11] takes the nonlinear nature of the distance-only measurements explicitly into account. Recently, in [12], the results in [11] have been extended to the case of ground vehicles processing mixtures of relative measurements (i.e., range and/or bearing) and a closed-form global optimal solution for the active-sensing problem is derived in the case of a single vehicle for arbitrary target-motion models. In addition, constraints on the maximum speed of the robots and on the minimum distance between the agents and the target have been incorporated into the problem formulation. However, the mathematical tools used by the authors in [11], [12], appear much more involved than those in [7], thus making their approach difficult to extend to different scenarios or to implement in a distributed fashion. Moreover, differently from [7]–[9], inter-agent cooperation does not seem fully exploited in [11], [12]. In [13] an approximate tracking algorithm is proposed, where the agents try to minimize the target’s location uncertainty using range-bearing sensors. The optimization problem (minimize the determinant of the target-position estimates’ covariance matrix), is solved separately by each agent by a greedy search over the discretized set of candidate headings. In [14] the authors have studied the optimal placement of range-only sensors for non-random static target position estimation. In this specific scenario, the determinant of the Fisher information matrix (the inverse of the covariance matrix) can be computed in closed-form and its critical points easily characterized. An optimal configuration, then turns out to be one in which the sensors are uniformly placed in a circular fashion around the target.

Other related work on cooperative active target tracking has been done in [22], where constant-speed unicycle robots are controlled using a distributed behavior-based approach, the communication takes place on a broadcast network and estimation is achieved by an unscened Kalman filter. However, differently from [11],[14] and [12], this paper does not provide an accurate theoretical characterization of the properties of the proposed coordination algorithm. Finally, in [23], the authors have studied the optimal routing of two camera-equipped fixed-wing aircraft cruising at fixed altitude, cooperatively tracking a single ground target. A perspective transformation relating the image-plane measurements to the ground allows to derive the geolocation error covariance matrix, and dynamic programming is used by the authors to compute optimal coordinated control policies which minimize the fused target localization error covariance.

Besides [7], [8] above, another source of inspiration for this paper came from some recent works in the cooperative localization literature [24]–[29]. In [24], [25], the authors have used the properties of the Riccati recursion arising from the extended Kalman filter to determine explicit upper bounds for the steady-state accuracy of the “Cooperative Localization” (CL) and “Simultaneous Cooperative Localization and Mapping” (C-SLAM) problems. The analysis in [24], [25] has been repeated in [26] for the “Cooperative Localization and Target Tracking” (CLATT) problem, and it has been shown that robots’ localization accuracy improves if the position of multiple moving targets is measured.

B. Original contributions and organization

This paper elaborates upon [7], [8] and extends them in several new directions.

First, by using a spherical coordinates representation, we present a generalization for a team of unmanned aerial vehicles modeled as double integrators and equipped with 3-D range-finding sensors.

Second, we introduce a gradient-based control design procedure that encompasses the three major optimum experimental design criteria. Our estimation mechanism leverages two different approaches: a cooperative one in which the vehicles’ target position measurements and covariance matrices of the measurement noise are fused together, and a noncooperative approach that relies on tools of dynamic noncooperative game theory. In the first scenario a Kalman filter is used for the fusion process, while in the second scenario the individual target’s position estimates are produced by coupled Kalman filter-like equations.

Third, in the case that the position of the vehicles is not perfectly known, we introduce a new problem, termed “Active Cooperative Localization and Multi-target Tracking” (ACLMT). In this problem, the aerial vehicles (modeled here as first-order systems) move in the 3-D space along paths that maximize both the accuracy of their own position estimate and that of multiple moving targets. Taking inspiration from [25], [26], analytical lower and upper bounds on the targets’ and vehicles’ positioning uncertainty are derived for ACLMT by exploiting the monotonicity property of the Riccati differential equation arising from the Kalman-Bucy filter.
These bounds are attractive since they allow us to study the impact of sensors’ accuracy and targets’ dynamics on the performance of our coordination strategy. Note that the active-sensing problem’s formulation considered in [11], [12] does not lend itself to the insightful performance analysis conducted in this work, and that ACLMT is, in several respects, more challenging than C-SLAM and CLATT: in fact, the targets’ motion in [26] is governed by a simple zero-velocity model (a random walk), and the ground robots in [24]–[26] move in open loop (i.e., no control feedback).

It is also worth emphasizing herein that in this paper we are not specifically concerned with the problem of designing distributed control laws for the aerial vehicles: in fact, one can straightforwardly move in this direction, by adapting the distributed strategies proposed in [8], [9] to our specific setup. Moreover, in order to keep our presentation accessible and concise, we decided not to incorporate collision-avoidance or maximum-speed constraints in our problem formulation. These issues will be addressed in future, more experiment-oriented works.

This article is based in part on previous material presented in [30], compared to which, we provide herein a more comprehensive literature review, new theoretical results and a more extensive numerical validation.

The rest of the paper is organized as follows. Sect. II presents the measurement model and estimation-and-control strategies for the aerial vehicles, both in the cooperative and noncooperative scenarios. In Sect. III, the ACLMT problem is introduced, and in Sect. IV the performance of ACLMT is explored. Finally, in Sect. V the results of extensive simulation experiments are discussed, and in Sect. VI the main contributions of the paper are summarized and possible avenues of future research are outlined.

II. ACTIVE TARGET TRACKING

A. Measurement model

Consider a team of $n$ aerial vehicles (hereafter, also simply “agents”) with positions $p_1, \ldots, p_n \in \mathbb{R}^3$ expressed with respect to a common global reference frame, and a target moving in 3-D according to the following model,

$$\dot{x}(t) = F x(t) + G u_f(t) + w(t),$$

where $x(t) \in \mathbb{R}^3$ denotes the position of the target at time $t$, $u_f(t) \in \mathbb{R}^3$ is an exogenous input and $w(t) \in \mathbb{R}^3$ is a continuous-time white Gaussian noise with zero mean and covariance matrix $Q \in \mathbb{R}^{3 \times 3}$. The observation $z_i(t) \in \mathbb{R}^3$ of the target’s position made by the $i$-th vehicle at time $t$ is assumed to be given by the following measurement model,

$$z_i(t) = H_i x(t) + v_i(t),$$

where $v_i(t)$ is a continuous-time zero-mean white Gaussian noise. In the following, we will assume that the measurement noises of the vehicles are independent and that the position $p_i$ of vehicle $i$ is perfectly known, e.g., from accurate GPS measurements (this last assumption will be relaxed in Sect. III). In addition, we will suppose that each agent is equipped with a magnetic compass (or an equivalent sensor), which provides its absolute orientation with respect to the fixed global reference frame: in this way, without loss of generality, we can assume that the reference frames attached to the vehicles are all aligned.

By adopting a standard 3-D range-finding sensor model [31, Sect. 4.3.2], we have that $H_i = I_3$ (the $3 \times 3$ identity matrix) and that the covariance matrix $R_{\text{rei}}(t) \in \mathbb{R}^{3 \times 3}$ of $v_i(t) \in \mathbb{R}^3$ assumes the form,

$$R_{\text{rei}}(t) = T_i(t) R(t) T_i^T(t),$$

where the rotation matrix $T_i$ (dropping the time index $t$), is given by,

$$T_i = R_z(\theta_i) R_y(\phi_i) = \begin{bmatrix} \cos \theta_i \cos \phi_i & -\sin \theta_i \cos \phi_i & \sin \phi_i \\ \sin \theta_i \cos \phi_i & \cos \theta_i \cos \phi_i & \sin \phi_i \\ -\sin \phi_i & 0 & \cos \phi_i \end{bmatrix},$$

and $R_z(\theta_i), R_y(\phi_i)$ denote the basic $3 \times 3$ rotation matrices about the $z$- and $y$-axes of an angle $\theta_i$ and $\phi_i$, respectively (see Fig. 1). $R_i$ in (2) is the covariance matrix of the measurement noise in the range-bearing-polar frame of vehicle $i$, and it has the following diagonal structure,

$$R_i = \text{diag}(\sigma_{\phi_1}^2, \sigma_{\phi_1}^2, \sigma_{\phi_1}^2).$$

(4)

The variance of the range-measurement noise $\sigma_{\phi_1}^2$ is typically represented by a function $f_r(r_i)$ of the Euclidean distance $r_i \triangleq ||p_i - x||_2$ from agent $i$ to the target [7]: the polar and bearing measurement-noise variances $\sigma_{\phi_1}^2$ and $\sigma_{\phi_1}^2$ can also be modeled as dependent on the range $r_i$ through the functions $f_\phi(r_i)$ and $f_\theta(r_i)$, respectively. In this paper, we will consider the following simple form for these functions,

$$f_r(r_i) \triangleq \alpha_2 (r_i - \alpha_1)^2 + \alpha_0,$$

$$f_\phi(r_i) \triangleq \alpha_\phi f_r(r_i),$$

$$f_\theta(r_i) \triangleq \alpha_\theta f_r(r_i),$$

Fig. 1. Measurement model for the 3-D range-finding sensor of vehicle $i$. The spherical coordinates $(r_i, \theta_i, \phi_i)$ have been used, where $r_i \in (0, +\infty)$ is the distance from vehicle $i$ to the target, $\theta_i \in (0, 2\pi) \setminus \{\pi\}$ is the bearing angle and $\phi_i \in (0, \pi)$ is the polar angle. $\{e_x, e_y, e_z\}$, $\{e_\theta, e_\phi, e_\psi\}$ represent the Cartesian and spherical orthonormal bases, respectively.
where \( a_0, a_1, a_2, a_\theta \) and \( a_\phi \) are strictly positive parameters (however, note that our subsequent results can be straightforwardly extended to the case of general differentiable positive-valued convex functions \( f_\phi(r_i), f_\theta(r_i), f_\phi(r_i) \)). This model assumes the existence of a “sweet spot” located at a distance \( a_1 \) from the target, where uncertainty in measurements is minimal (see Fig. 2). Parameter \( a_1 \) depends on the physics of the sensing device, and it may vary from several tens of centimeters for infrared sensors, to few meters for time-of-flight sensors (e.g. ultrasonic sensors). Although environmental factors, such as temperature, humidity or background noise, might affect the sweet spot, \( a_1 \) is typically assumed to be a constant parameter in the literature. It is finally worth observing that \( a_1 \), together with the other parameters \( a_0, a_2, a_\theta, a_\phi \), can be estimated from real data collected by the range-finding sensor, using standard least-squares methods (c.f., e.g. [32]).

Remark 1: Note that the transformation (2) from spherical coordinates, \( R_i \), to Cartesian coordinates\(^1\), \( R_i^{Car} \), is valid only for small \( \theta_i \) and \( \phi_i \) (c.f. [4, Sect. 10.4.3]). However, extensive simulation experiments (see Sect. V), have shown that a violation of this “small-angle condition”, does not lead to a significant performance degradation of the target-tracking algorithms proposed in this paper.

In Sects. II-B and II-C we will present a cooperative estimation-and-control strategy. In Sect. II-D, instead, we will briefly describe a noncooperative approach that leverages notions of differential game theory, and that is of value in throwing further light on the active target-tracking problem.

B. Cooperative control

The target-position measurements \( z_i \) and covariances \( R_i^{Car} \) from the \( n \) agents will be fused together to obtain a global target position estimate \( \hat{X}_{fus} \) and global position-error covariance \( P_{fus} \) (see Sect. II-C). In this section, we will present a cooperative control strategy that moves the aerial vehicles in order to minimize a scalar function of the covariance \( P_{fus} \), thus reducing the target’s localization error (c.f. [8]).

In order to show the generality of our approach, for the design of the gradient-based control laws we will simultaneously consider the cost functions for three most popular optimum experimental design criteria [6, Sect. 2.3],

\[
J = \ln \det(P_{fus}) ,
\]

for the D-optimality (determinant) criterion,

\[
J = \text{tr}(P_{fus}) ,
\]

for the A-optimality (trace) criterion, and

\[
J = \lambda_{\max}(P_{fus}) = \| P_{fus} \|_2 ,
\]

for the E-optimality criterion, where \( \lambda_{\max}(P_{fus}) \) denotes the maximum eigenvalue of \( P_{fus} \) and \( \| P_{fus} \|_2 \) is the spectral norm of \( P_{fus} \). The D-optimum design (the most widely used criterion), minimizes the volume of the uncertainty ellipsoid for the estimates, the E-optimum design minimizes the length of the largest axis of the same ellipsoid, and the A-optimum design suppresses the average variance of the estimates. Note that although the cost functions (5)-(7) are convex with respect to the relative position and orientation of the vehicles and target, and they are thus prone to local minima [33].

Since the cost function (7) is not differentiable in general (in fact, the gradient of \( \lambda_{\max}(P_{fus}) \) does not exist when the maximum eigenvalue of \( P_{fus} \) is not simple), in the following we will consider its smoothed version [34]:

\[
\Phi_\varepsilon(P_{fus}) = \varepsilon \ln \left( \sum_{i=1}^{3} \exp \left( \lambda_i(P_{fus})/\varepsilon \right) \right) ,
\]

where \( \varepsilon > 0 \) is an assigned smoothing parameter and \( \lambda_i(P_{fus}) \) denotes the \( i \)-th eigenvalue of \( P_{fus} \). Note that function \( \Phi_\varepsilon(P_{fus}) \) is of class \( C^\infty \) and it possesses the following uniform approximation property to \( \lambda_{\max}(P_{fus}) \).

\[
0 \leq \Phi_\varepsilon(P_{fus}) - \lambda_{\max}(P_{fus}) \leq \varepsilon \ln(3) , \quad \forall \varepsilon > 0 ,
\]

from which it follows that \( \lim_{\varepsilon \rightarrow 0} \Phi_\varepsilon(P_{fus}) = \lambda_{\max}(P_{fus}) \). As indicated in [35, p. 249], it turns out that when \( \varepsilon \in [10^{-4}, 10^{-6}] \), \( \Phi_\varepsilon(P_{fus}) \) yields an excellent approximation to \( \lambda_{\max}(P_{fus}) \).

With the exception of Sects. III and IV, the following double-integrator model will be used for the aerial vehicles,

\[
\dot{p}_i = q_i , \quad \dot{q}_i = u_i , \quad i \in \{1, \ldots, n\} ,
\]

where \( q_i \in \mathbb{R}^3 \) denotes the velocity of vehicle \( i \) and \( u_i \in \mathbb{R}^3 \) its control input. The gradient controller of vehicle \( i \) will be then of the form,

\[
u_i = -B q_i - \Gamma T_i \left[ \frac{1}{r_i} \frac{\partial J}{\partial \theta_i} , \frac{1}{r_i} \frac{\partial J}{\partial \phi_i} , \frac{\partial J}{\partial r_i} \right]^T ,
\]

where \( B \in \mathbb{R}^{3x3} , B \succeq 0 \), is a damping matrix and \( \Gamma \in \mathbb{R}^{3x3} , \Gamma \succeq 0 \), is a gain matrix\(^2\). In order to simplify the forthcoming

\(^1\)The spherical coordinates \( (r, \theta, \phi) \) are related to the Cartesian coordinates \( (x, y, z) \) by the nonlinear transformation \( r = \sqrt{x^2 + y^2 + z^2} , \theta = \text{atan2}(y, x) , \phi = \text{acos}(z/r) \), where \( \text{atan2}(\cdot, \cdot) \) stands for the four-quadrant inverse tangent.

\(^2\)The symbols “\( \succ \)”,” “\( \succeq \)”, denote the matrix inequality in the positive definite and positive semidefinite sense, respectively.
derivations, let us introduce the following $3 \times 3$ matrices, whose explicit expression will be given in Sect II-C,

$$
P_i^f \triangleq \frac{\partial P_{\text{fus}}}{\partial r_i}, \quad P_i^{\phi} \triangleq \frac{\partial P_{\text{fus}}}{\partial \theta_i}, \quad P_i^{\phi} \triangleq \frac{\partial P_{\text{fus}}}{\partial \phi_i}. \quad (10)$$

By using [6, Th. B.17], we can explicitly compute the partial derivatives appearing on the right-hand side of (9) for the D-optimality criterion,

$$
\frac{\partial J}{\partial r_i} = \text{tr}(P_i^{\text{fus}} P_i^{f}), \quad \frac{\partial J}{\partial \theta_i} = \text{tr}(P_i^{\text{fus}} P_i^{\phi}), \quad \frac{\partial J}{\partial \phi_i} = \text{tr}(P_i^{\text{fus}} P_i^{\phi}),
$$

for the A-optimality criterion,

$$
\frac{\partial J}{\partial r_i} = \text{tr}(U^T \text{diag}(\nabla \Phi_{\text{fus}}(P_{\text{fus}})) U P_i^{f}), \quad \frac{\partial J}{\partial \theta_i} = \text{tr}(U^T \text{diag}(\nabla \Phi_{\text{fus}}(P_{\text{fus}})) U P_i^{\phi}), \quad \frac{\partial J}{\partial \phi_i} = \text{tr}(U^T \text{diag}(\nabla \Phi_{\text{fus}}(P_{\text{fus}})) U P_i^{\phi}).
$$

(11)

The $l$-th component of the gradient vector $\nabla \Phi_{\text{fus}}(P_{\text{fus}})$ in (11) is,

$$
[\nabla \Phi_{\text{fus}}(P_{\text{fus}})]_l = \frac{\exp(\lambda_l(P_{\text{fus}})/\varepsilon)}{3} \sum_{j=1}^{3} \exp(\lambda_j(P_{\text{fus}})/\varepsilon), \quad l \in \{1, 2, 3\},
$$

and the orthogonal matrix $U$ satisfies $P_{\text{fus}} = U^T \text{diag}(\lambda(P_{\text{fus}})) U$, where $\text{diag}(\lambda(P_{\text{fus}}))$ is a diagonal matrix with the elements of the vector $\lambda(P_{\text{fus}}) = [\lambda_1(P_{\text{fus}}), \lambda_2(P_{\text{fus}}), \lambda_3(P_{\text{fus}})]^T$. Let $\lambda_1(P_{\text{fus}}) \geq \lambda_2(P_{\text{fus}}) \geq \lambda_3(P_{\text{fus}})$, put on its main diagonal.

Remark 2: Note that the controller (9) is centralized since $P_{\text{fus}}, P_i^{f}, P_i^{\phi}$ and $P_i^{\phi}$ contain information from all the agents. However, the control law (9) can be implemented in a distributed fashion by replacing any unavailable global quantity with local estimates, as done, e.g., in [8], [9].

C. Kalman-filter fusion

As anticipated in Sect. II-B, we describe here a method inspired by [8], to fuse the local target position measurements and error covariance matrices: in this way, we can determine the three matrices in (10), necessary for the implementation of controller (9). The method defines $\hat{X}_{\text{fus}}$ and $P_{\text{fus}}$ by means of a Kalman-Bucy filter [36, Sect. 4.5]:

$$
\dot{\hat{P}}_{\text{fus}} = FP_{\text{fus}} + P_{\text{fus}} F^T + Q - P_{\text{fus}} C \hat{C}_{\text{fus}}, \quad (12)
$$

$$
\dot{\hat{X}}_{\text{fus}} = F \hat{X}_{\text{fus}} + G u_T + P_{\text{fus}} (y - C \hat{X}_{\text{fus}}), \quad (13)
$$

where

$$
P = \sum_{i=1}^{n} (R_i^{\text{Car}})^{-1}, \quad \hat{y} = \sum_{i=1}^{n} (R_i^{\text{Car}})^{-1} z_i,
$$

and (12) and (13) are respectively initialized with,

$$
\hat{P}_{\text{fus}}(t_0) = \left[ \sum_{i=1}^{n} (R_i^{\text{Car}}(t_0))^{-1} \right]^{-1}, \quad (14)
$$

$$
\hat{X}_{\text{fus}}(t_0) = \hat{P}_{\text{fus}}(t_0) \sum_{i=1}^{n} (R_i^{\text{Car}}(t_0))^{-1} z_i(t_0), \quad (15)
$$

where $t_0$ denotes the initial time instant. Note that (15) is a weighted least-squares estimate for the position of the target, and that (14) is the covariance matrix of the weighted least-squares estimator. By taking the partial derivatives with respect to $r_i, \theta_i$ and $\phi_i$ on both sides of the Riccati differential equation (RDE) (12), we get the following three Lyapunov differential equations,

$$
\dot{P}_i^f = (F - P_{\text{fus}} C) P_i^f + P_i^f (F - P_{\text{fus}} C)^T
$$

$$+ 2 \alpha_2 (r_i - a_i) P_{\text{fus}} T_i R_i^{-1} \text{diag}(\alpha_0, \alpha_0, 1) T_i^T P_{\text{fus}},
$$

$$
\dot{P}_i^\phi = (F - P_{\text{fus}} C) P_i^\phi + P_i^\phi (F - P_{\text{fus}} C)^T
$$

$$- P_{\text{fus}} (A_{\phi_i} + A_{\phi_i}^T) P_{\text{fus}}, \quad A_{\phi_i} = \Theta T_i R_i^{-1} T_i^T,
$$

$$
\dot{P}_i^{\phi} = (F - P_{\text{fus}} C) P_i^{\phi} + P_i^{\phi} (F - P_{\text{fus}} C)^T
$$

$$- P_{\text{fus}} (A_{\phi_i} + A_{\phi_i}^T) P_{\text{fus}}, \quad A_{\phi_i} = T_i R_i^{-1} T_i^T,
$$

(16)

where

$$
\Theta = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \Upsilon = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix}.
$$

The equations in (16) are respectively initialized with,

$$
P_i^f(t_0) = 2 \alpha_2 (r_i(t_0) - a_i) P_{\text{fus}}(t_0) T_i(t_0) R_i^{-2}(t_0) \text{diag}(\alpha_0, \alpha_0, 1) T_i^T P_{\text{fus}}(t_0),
$$

$$
P_i^\phi(t_0) = - P_{\text{fus}}(t_0) (A_{\phi_i}(t_0) + A_{\phi_i}^T(t_0)) P_{\text{fus}}(t_0),
$$

$$
P_i^{\phi}(t_0) = - P_{\text{fus}}(t_0) (A_{\phi_i}(t_0) + A_{\phi_i}^T(t_0)) P_{\text{fus}}(t_0).
$$

D. Noncooperative estimation and control

In Sects. II-B and II-C, we have assumed that the $n$ aerial vehicles coordinate their 3-D motion in order to estimate the position of the target as accurately as possible. In this section, we briefly explore the converse scenario (of potential interest, e.g., in a military setting) in which antagonistic agents compete which each other in order to optimally estimate the location of the moving target. The control strategy obtained in this case is more amenable to a decentralized implementation than that presented in Sect. II-B, but as it is natural to expect, it leads to less accurate individual estimates of the target’s position (cf. Figs. 4 and 5). More specifically, in this section we re-interpret the active-target tracking problem as an $n$-person noncooperative nonzero-sum differential game [37, Sect. 6.1], and we are interested in the existence of possible feedback Nash equilibrium strategies [38]. As shown in Corollary A-1 of [39], for the class of $n$-person linear-quadratic differential games, a set of these equilibrium strategies can be expressed
in closed-form. The dualization of equation (A-16) of this corollary (with zero cross-weighting matrices), leads to the following Kalman filter-like equations for \( i \in \{1, \ldots, n\} \),

\[
\dot{\mathbf{x}}_i = \mathbf{F}_i \mathbf{x}_i + \mathbf{G}_i \mathbf{u}_T + \mathbf{P}_i (\mathbf{R}_i^{-1} - \mathbf{z}_i - \hat{\mathbf{x}}_i),
\]

where (17) are \( n \) coupled RDEs, and \( \hat{\mathbf{x}}_i \) and \( \mathbf{P}_i \) denote the target’s position estimate of agent \( i \) and the associated error covariance matrix, respectively. Similarly to Sect. II-C, we can initialize equations (17) and (18) with \( \mathbf{P}_i(t_0) = \mathbf{R}_i^{-1}(t_0) \) and \( \hat{\mathbf{x}}_i(t_0) = \mathbf{P}_i(t_0)(\mathbf{R}_i^{-1}(t_0))^{-1} \mathbf{z}_i(t_0) \), respectively. Note that Lemma 6.6.1 of [37], ensures the existence of positive-semidefinite solutions to (17).

In the noncooperative framework described in this section, we can define the gradient controller for agent \( i \) as

\[
\mathbf{u}_i = -\mathbf{B}_i \mathbf{q}_i - \Gamma_i \left[ \frac{1}{r_i} \frac{\partial J_i}{\partial x_i} + \frac{1}{r_i} \frac{\partial J_i}{\partial \phi_i} \right] T \mathbf{P}_i^{-1} \mathbf{P}_i^{T},
\]

where \( J_i = \ln \det (\mathbf{P}_i) \), or alternatively \( J_i = \text{tr}(\mathbf{P}_i) \), \( J_i = \lambda_{\text{max}}(\mathbf{P}_i) \) (c.f. equation (9)). In order to write concise equations in the remaining of this section, let us introduce (with a slight abuse of notation), the following \( 3 \times 3 \) matrices for all \( i, k \in \{1, \ldots, n\} \),

\[
\mathbf{P}_{ik}^0 \triangleq \frac{\partial \mathbf{P}_i}{\partial r_k}, \quad \mathbf{P}_{ik}^0 \triangleq \frac{\partial \mathbf{P}_i}{\partial \theta_k}, \quad \mathbf{P}_{ik}^0 \triangleq \frac{\partial \mathbf{P}_i}{\partial \phi_k}.
\]

By using [6, Th. B.17], we can compute, as before, the partial derivatives appearing on the right-hand side of (19), for the D-, A- and E-optimality criteria. For the sake of brevity, we only report here the explicit expressions for the first criterion,

\[
\frac{\partial J_i}{\partial r_i} = \text{tr}(\mathbf{P}_i^{-1} \mathbf{P}_{ii}^0), \quad \frac{\partial J_i}{\partial \theta_i} = \text{tr}(\mathbf{P}_i^{-1} \mathbf{P}_{ti}^0), \quad \frac{\partial J_i}{\partial \phi_i} = \text{tr}(\mathbf{P}_i^{-1} \mathbf{P}_{pi}^0),
\]

being the others easily deducible from the equations in Sect. II-B. The matrix \( \mathbf{P}_{ii}^0 \) in (20), is obtained by integrating the following \( n^2 \) coupled Lyapunov differential equations for \( i, k \in \{1, \ldots, n\} \),

\[
\dot{\mathbf{P}}_{ik} = \mathbf{E} \mathbf{P}_{ik}^T + \mathbf{P}_{ik}^T \mathbf{E}^T - \sum_{j=1}^{n} \left( \mathbf{W}_{ij} \mathbf{P}_{jk} + \mathbf{P}_{jk} \mathbf{W}_{ij}^T \right) - \Omega_{ik},
\]

where

\[
\mathbf{E} \triangleq \mathbf{F} - \sum_{j=1}^{n} \mathbf{P}_{ij} (\mathbf{R}_j^{-1}), \quad \mathbf{W}_{ij} \triangleq \mathbf{P}_{ij} (\mathbf{R}_i^{-1}), \quad \Omega_{ik} \triangleq \rho_{ik} \left[ \mathbf{P}_i \frac{\partial (\mathbf{R}_i^{-1})}{\partial r_k} \mathbf{P}_k + \mathbf{P}_k \frac{\partial (\mathbf{R}_i^{-1})}{\partial r_k} \mathbf{P}_i \right],
\]

\[
\frac{\partial (\mathbf{R}_i^{-1})}{\partial r_k} = -2 \alpha_2 (r_k - a_i) \mathbf{T}_k \mathbf{R}_k^{-2} \text{diag}(\alpha_\phi, \alpha_\theta, 1) \mathbf{T}_k^T,
\]

with \( \rho_{ik} = 1/2 \) if \( i = k \), and \( \rho_{ik} = 1 \) otherwise. Analogously, \( \mathbf{P}_{ii}^0 \) and \( \mathbf{P}_{ii}^0 \) in (20) are obtained from the integration of the following coupled Lyapunov differential equations for \( i, k \in \{1, \ldots, n\} \),

\[
\dot{\mathbf{P}}_{ik} = \mathbf{E} \mathbf{P}_{ik}^T + \mathbf{P}_{ik}^T \mathbf{E}^T - \sum_{j=1, j \neq i}^{n} \left( \mathbf{W}_{ij} \mathbf{P}_{jk}^\theta + \mathbf{P}_{jk}^\theta \mathbf{W}_{ij}^T \right) - \rho_{ik} \left[ \mathbf{P}_i (\mathbf{A}_{\theta_ik} + \mathbf{A}_{\phi_ik}^T) \mathbf{P}_k + \mathbf{P}_k (\mathbf{A}_{\theta_ik} + \mathbf{A}_{\phi_ik}^T) \mathbf{P}_i \right],
\]

where \( \mathbf{A}_{\theta_ik} \) and \( \mathbf{A}_{\phi_ik} \) are defined as in (16), and (21)-(22) are initialized analogously to the equations in (16).

III. ACTIVE COOPERATIVE LOCALIZATION AND MULTI-TARGET TRACKING (ACMLT)

In this section, we extend the cooperative scenario described in Sects. II-B and II-C to one in which we have \( m \) moving targets with positions \( x_1, \ldots, x_m \) and \( n \) aerial vehicles whose location \( p_1, \ldots, p_n \) is not exactly known. The agents simultaneously perform vehicle-to-target and vehicle-to-vehicle relative measurements, and they move along trajectories that maximize the accuracy of both their own and the other targets’/vehicles’ position estimate. We will refer to this new problem as “Active Cooperative Localization and Multi-target Tracking” (ACMLT). In what follows, we will assume that each vehicle is capable of distinguishing the other agents/targets. The “assignment information” (i.e., which vehicle is assigned to which vehicle/target) is incorporated into the association matrix \( \mathbf{A} \), that, as we will see later in this section, is instrumental in defining the control for the \( n \) vehicles.

**Definition 1 (Association matrix \( \mathbf{A} \)):**

Let \( \mathbf{x}_{tot} \triangleq [x_1^T, \ldots, x_m^T, p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{(m+n)} \) be the stacked vector of the positions of the \( m \) targets and \( n \) vehicles. Let \( \mathcal{G}(\mathbf{x}_{tot}) = (\mathcal{V}, \mathcal{T} \cup \mathcal{U}, \mathcal{E}(\mathbf{x}_{tot})) \) be a bipartite dynamic digraph with vertex sets \( \mathcal{V} \) and \( \mathcal{T} \cup \mathcal{U} \) (\( \mathcal{V} = \{v_1, \ldots, v_n\} \) is the set of vehicles and \( \mathcal{T} = \{t_1, \ldots, t_m\} \) the set of targets), and edge set \( \mathcal{E}(\mathbf{x}_{tot}) \) (obviously, \( \{v_i, v_j\} \notin \mathcal{E}(\mathbf{x}_{tot}) \) for all \( i \) and \( t \)). We define the association matrix of \( \mathcal{G}(\mathbf{x}_{tot}) \) (see the illustrative example in Fig. 3), to be the \( n \times (m + n) \) matrix \( \mathbf{A}(\mathbf{x}_{tot}) = [\ell_{ih}(\mathbf{x}_{tot})] \), 

\[
\sum_{h=1}^{n} \ell_{ih}(\mathbf{x}_{tot}) = 1, \quad \text{such that } \ell_{ih}(\mathbf{x}_{tot}) > 0 \text{ if } \{v_i, t_\mathcal{U} \cup \mathcal{V} \} \in \mathcal{E}(\mathbf{x}_{tot}) \text{ and } \ell_{ih}(\mathbf{x}_{tot}) = 0 \text{ otherwise, } i \in \{1, \ldots, n\}, h \in \{1, \ldots, m \}, \text{where } [t_\mathcal{U} \cup \mathcal{V} ]_h \text{ denotes the } h \text{-th element of the set } \mathcal{T} \cup \mathcal{U}.\]

We will assume that \( \ell_{ih}(\mathbf{x}_{tot}) = 0 \) when vehicle \( i \) is not able to sense the \( h \)-th target/vehicle at time \( t \). On the other hand, the value of the positive weights \( \ell_{ih}(\mathbf{x}_{tot}) \) may vary according to the distance of vehicle \( i \) to the \( h \)-th target/vehicle or it may be chosen according to other priority/hierarchical criteria, depending on the specific application scenario.

By extending equation (1), let us suppose that the motion of the targets is governed by,

\[
\dot{x}_j = \mathbf{F}_j \mathbf{x}_j + \mathbf{G}_j \mathbf{u}_T + \mathbf{w}_j, \quad j \in \{1, \ldots, m\},
\]
The measurement model of agent besides their position, which is impractical in a real application.

\[ \Lambda(x_{tot}) = \begin{bmatrix} T_1 & T_2 & V_1 & V_2 & V_3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Fig. 3. Example of association matrix: (top) bipartite digraph \( G(x_{tot}) \) in the case of three vehicles \( \{V_1, V_2, V_3\} \) and two targets \( \{T_1, T_2\} \), and (bottom) corresponding \( 3 \times 5 \) association matrix \( \Lambda(x_{tot}) \).

Different from the preceding section, we will suppose here that the motion of the aerial vehicles is governed by the following first-order model,

\[ \dot{p}_i = u_i + n_i, \quad i \in \{1, \ldots, n\}, \quad (24) \]

where \( p_i \in \mathbb{R}^3 \) is the position of agent \( i \), \( u_i \in \mathbb{R}^3 \) its control input, and \( n_i \in \mathbb{R}^3 \) is a continuous-time white Gaussian noise with zero mean and diagonal covariance matrix \( Q_i \in \mathbb{R}^{3 \times 3} \).

By combining equations (23) and (24) together, we obtain the following system,

\[ x_{tot} = F_{tot} x_{tot} + G_{tot} u_{tot} + w_{tot} \]

\[ \triangleq \text{blkdiag}(F_1, \ldots, F_m, 0_{3 \times 3}, \ldots, 0_{3 \times 3}) x_{tot} + \text{blkdiag}(G_1, \ldots, G_m, I_3, \ldots, I_3) u_{tot} + w_{tot}, \]

where \( u_{tot} \triangleq [u_{1T}^T, \ldots, u_{mT}^T, u_{1S}^T, \ldots, u_{mS}^T]^T \in \mathbb{R}^{3(m+n)} \),

\[ w_{tot} \triangleq \begin{bmatrix} w_{1T}^T, \ldots, w_{mT}^T, w_{1S}^T, \ldots, w_{mS}^T \end{bmatrix}^T \in \mathbb{R}^{3(m+n)} \),

\( \text{blkdiag}(. \) denotes a block diagonal matrix obtained by the concatenation of its matrix arguments, and \( 0_{3 \times 3} \) is the \( 3 \times 3 \) matrix of zeros.

As in Sect. II-A, we assume here that vehicle \( i \) is equipped with a 3-D range-finding sensor: this time, however, we will suppose that the position \( p_i \) of agent \( i \) is not perfectly known. The measurement model of agent \( i \) will then take the form,

\[ z_i = x_{tot} + v_i + d_i, \quad i \in \{1, \ldots, n\}, \quad (26) \]

where the covariance matrix \( R_{Car}^{i} \) of the vehicle-to-target/vehicle-to-vehicle measurement noise \( v_i \in \mathbb{R}^{l(m+n)} \) is again of the form (2), but with,

\[ R_i = \text{blkdiag}(R_{Vi}^{1}, \ldots, R_{Vi}^{m}, R_{Vi}^{1}, \ldots, R_{Vi}^{n}), \]

and

\[ T_i = \text{blkdiag}(T_{Vi}^{1}, \ldots, T_{Vi}^{m}, T_{Vi}^{1}, \ldots, T_{Vi}^{n}), \]

where \( R_{Vi}^{mj}, R_{Vi}^{kj} \in \mathbb{R}^{3 \times 3} \) are covariance matrices of the form (4) and \( T_{Vi}^{mj}, T_{Vi}^{kj} \in \mathbb{R}^{3 \times 3} \) are rotation matrices of the form (3), relative to vehicle \( i \) with respect to target \( j \in \{1, \ldots, m\} \) and vehicle \( k \in \{1, \ldots, n\} \), respectively. Note that, in particular, \( R_{Vi} = 0_{3 \times 3} \) and \( T_{Vi} = I_3 \). The zero-mean white Gaussian noise \( d_i \in \mathbb{R}^{l(m+n)} \) models the uncertainty on the self-localization of vehicle \( i \) and its covariance matrix is,

\[ \Delta_i = \text{blkdiag}(D_1, \ldots, D_1), \]

where \( D_1 \) is a \( 3 \times 3 \) diagonal matrix. The noises \( v_i \) and \( d_i \) are assumed to be independent. The fusion method described in Sect. II-C is still valid in the more challenging scenario studied in this section as long as suitable modifications are done. In particular, the equations of the Kalman-Bucy filter for system (25)-(26), are now:

\[ P_{tot} = F_{tot} P_{ fus } + P_{ fus } F_{tot}^T + Q_{tot} - P_{ fus } C_{tot} P_{ fus }, \quad (27) \]

\[ \hat{x}_{tot} = F_{tot} \hat{x}_{tot} + G_{tot} u_{tot} + P_{ fus } (y_{tot} - C_{tot} \hat{x}_{tot}), \quad (28) \]

where

\[ C_{tot} \triangleq \sum_{i=1}^{n} (R_{Car}^{i} + \Delta_i)^{-1}, \quad y_{tot} \triangleq \sum_{i=1}^{n} (R_{Car}^{i} + \Delta_i)^{-1} z_i, \]

\[ Q_{tot} \triangleq \text{blkdiag}(Q_1, \ldots, Q_m, N_1, \ldots, N_n). \]

Equations (27)-(28) are initialized with (14)-(15) where \( R_{Car}^{i} (t_0) \) is replaced with \( R_{Car}^{i} (t_0) + \Delta_i \). Let now \( u_{ij} \) denote the gradient control of vehicle \( i \) due to the influence of the \( h \)-th vehicle/target,

\[ u_{ij} = -\Gamma T_{ij} [1/r_{ij} \sin \theta_{ij} | \partial J/ \partial \theta_{ij} | 1/r_{ij} \sin \theta_{ij} | \partial J/ \partial r_{ij} | \partial J/ \partial r_{ij} ]^T, \]

where the symbols \( T_{ij}, r_{ij}, \theta_{ij} \) and \( \phi_{ij} \) have an obvious meaning (c.f. (9) and (24)). Then, according to Def. 1, we can write the (overall) control \( u_i \) of vehicle \( i \), as (c.f. [8]):

\[ u_i \triangleq \sum_{h=1}^{m+n} \ell_{ih} (x_{tot}) u_{ih}, \quad i \in \{1, \ldots, n\}. \]

IV. PERFORMANCE OF ACLMT

In this section we will derive analytical bounds on the covariance matrix \( P_{ fus } \) of the Kalman-Bucy filter for system (25)-(26), that will allow us to study the role played by the sensors’ accuracy and targets’ dynamics on the performance of ACLMT.

In order to state the two main results of this section (Prop. 1 and Prop. 2), we briefly review here an interesting property of
Hermitian Riccati differential equations (HRDEs) [37, Ch. 2]. Consider the following RDE,

\[ \dot{P} = -A^*(t)P - PA(t) - M(t) + PS(t)P, \quad t \in \mathcal{I}, \quad (29) \]

where \( A^*(t) \) is the conjugate transpose of \( A(t) \in \mathbb{C}^{q \times q} \), and \( M(t), S(t) \) are \( q \times q \) Hermitian matrices for \( t \in \mathcal{I} \subseteq \mathbb{R}_{\geq 0} \). The coefficients \( A(t), M(t) \) and \( S(t) \) are assumed to be piecewise continuous, locally bounded functions of time. Equation (29) is called HRDE since its solutions with Hermitian initial values \( P(t_0) \), are Hermitian.

The following comparison theorem [37, Th. 4.1.4], shows that Hermitian solutions of HRDEs are monotonically dependent on initial values and on the coefficients: in other words, the solutions of HRDEs tend to preserve their ordering with respect to positive definiteness as their argument changes.

**Theorem 1 (Comparison theorem):** Let \( \mathcal{I} \subseteq \mathbb{R}_{\geq 0} \) be some interval, \( t_0 \in \mathcal{I} \), and let the Hamiltonian matrices\(^4\),

\[ \begin{bmatrix} A_j(t) & -S_j(t) \\ -M_j(t) & -A_j^*(t) \end{bmatrix}, \quad j \in \{1, 2\}, \]

be piecewise continuous and locally bounded on \( \mathcal{I} \), with \( S_j(t) \) and \( M_j(t) \) Hermitian matrices for \( t \in \mathcal{I} \). If \( P_j, j \in \{1, 2\} \), are on \( \mathcal{I} \) solutions of

\[ \dot{P}_j = -A_j^*(t)P_j - P_jA_j(t) - M_j(t) + P_jS_j(t)P_j, \quad (30) \]

with \( P_1(t_0) \geq P_2(t_0) \) and \( P_1(t_0), P_2(t_0) \) Hermitian, then,

\[ \begin{bmatrix} M_1(t) & A_1^*(t) \\ A_1(t) & -S_1(t) \end{bmatrix} \preceq \begin{bmatrix} M_2(t) & A_2^*(t) \\ A_2(t) & -S_2(t) \end{bmatrix} \quad \text{for} \quad t \in \mathcal{I}, \quad (31) \]

implies \( P_1(t) \geq P_2(t) \) for \( t \in \mathcal{I} \cap [t_0, +\infty) \), i.e., the solutions of (30) depend monotonically on

\[ \begin{bmatrix} M_j(t) & A_j^*(t) \\ A_j(t) & -S_j(t) \end{bmatrix}, \quad j \in \{1, 2\}, \]

(and in particular on \( M_j(t) \) and \( -S_j(t) \)) and on the initial value \( P_j(t_0) \).

From Th. 1, we immediately deduce the following result concerning the existence of lower and upper bounds on the solution of the RDE (27).

**Proposition 1 (Bounds on \( P_{\text{fin}} \))**: Let \( \mathcal{I} \subseteq \mathbb{R}_{\geq 0} \) be some interval, and \( t_0 \in \mathcal{I} \). If \( P^L_{\text{fin}}, P_{\text{fin}} \) and \( P^U_{\text{fin}} \) are on \( \mathcal{I} \) solutions of the following RDEs,

\[ \begin{align*}
&\dot{P}^L_{\text{fin}} = F^L_{\text{fin}}P^L_{\text{fin}} + P^L_{\text{fin}}F^T_{\text{fin}} + Q_{\text{tot}} - P^L_{\text{fin}}C_{\text{tot}}^U P^L_{\text{fin}}, \quad (32) \\
&\dot{P}_{\text{fin}} = F_{\text{fin}}P_{\text{fin}} + P_{\text{fin}}F^T_{\text{fin}} + Q_{\text{tot}} - P_{\text{fin}}C_{\text{tot}}(t) P_{\text{fin}}, \quad (33) \\
&\dot{P}^U_{\text{fin}} = F^U_{\text{fin}}P^U_{\text{fin}} + P^U_{\text{fin}}F^T_{\text{fin}} + Q_{\text{tot}} - P^U_{\text{fin}}C_{\text{tot}}(t) P^U_{\text{fin}}, \quad (34)
\end{align*} \]

with \( P_{\text{fin}}(t_0) = P^L_{\text{fin}}(t_0) = P^U_{\text{fin}}(t_0) \), \( P^L_{\text{fin}}(t_0) \) symmetric, and \( C_{\text{tot}}, C_{\text{tot}}^U \) constant symmetric matrices, then,

\[ C_{\text{tot}} \preceq C_{\text{tot}}(t) \preceq C_{\text{tot}}^U \quad \text{for} \quad t \in \mathcal{I}, \]

implies

\[ P^L_{\text{fin}}(t) \preceq P_{\text{fin}}(t) \preceq P^U_{\text{fin}}(t) \quad \text{for} \quad t \in \mathcal{I} \cap [t_0, +\infty). \quad (35) \]

\[^4\text{A matrix} \quad \mathcal{H} \quad \text{in} \quad \mathbb{C}^{2q \times 2q} \text{is called Hamiltonian if} \quad \mathcal{J} \mathcal{H} = (\mathcal{J} \mathcal{H})^*, \quad \text{where} \quad \mathcal{J} \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{is the unit imaginary matrix} [37, Def. 2.1. (iii)].\]

**Proof**: The result can be proved by performing the following substitutions in the statement of Th. 1 (we consider herein only the first pair of equations (32)-(33); a similar reasoning applies to the second pair (33)-(34)):}
Agent can measure, determined, e.g., by the size of the 3-D navigation environment (c.f. [24]).

If we substitute the upper and lower bounds \( C_{\text{tot}}^\text{u}, C_{\text{tot}} \)

in (32), (34), and numerically evaluate the solution of the two equations, we obtain a lower and upper bound on \( P_{\text{fin}} \)
at any instant of time. Since the system under investigation is completely observable, then after undergoing an initial transient phase, \( P_{\text{fin}} \) will reach a steady state. It is therefore worth studying the steady-state behavior of the uncertainty over the targets’ and vehicles’ position: this information is conveyed by the asymptotic solutions \( P_{\text{fin}}^L(\infty), P_{\text{fin}}^U(\infty) \)
of equations (32), (34). Let the pair \((P_{\text{fin}}, C_{\text{fin}})\)
denotes either \((P_{\text{fin}}^L, C_{\text{fin}})\) or \((P_{\text{fin}}^U, C_{\text{fin}})\). From Kalman-filter theory [36, p. 476], we know that if the pair \((F_{\text{fin}}, Q_{\text{fin}}^{1/2})\) is stabilizable, then the solution of

\[
\dot{P}_{\text{fin}}(t) = F_{\text{tot}} P_{\text{fin}}(t) + P_{\text{fin}}(t) F_{\text{tot}}^T + Q_{\text{fin}} - P_{\text{fin}}(t) C_{\text{fin}} P_{\text{fin}}(t),
\]

converges to a unique positive-definite matrix at steady state from any initial condition \(P_{\text{fin}}(t_0)\), determined by solving the continuous-time algebraic Riccati equation (CARE).

\[
F_{\text{tot}} P_{\text{fin}}(\infty) + P_{\text{fin}}(\infty) F_{\text{tot}}^T + Q_{\text{fin}} - P_{\text{fin}}(\infty) C_{\text{fin}} P_{\text{fin}}(\infty) = 0.
\]

Therefore, in summary, the study of the steady-state performance of ACLM, simply translates into solving two standard CAREs for \(P_{\text{fin}}^L(\infty)\) and \(P_{\text{fin}}^U(\infty)\), for which efficient numerical routines exist in mathematical software packages.

Note that the bounds described in this section are relative to the uncertainty on the considered target and measurement models, and they are not bounds on the (unknown) globally optimal trajectory of the vehicles (c.f., [16]).

**Remark 3 (Bounds on the cost function J):** From condition (35) and Weyl’s inequality [40, Th. 8.4.9], it follows that for \(t \in I \cap [t_0, +\infty)\),

\[
\ln \det(P_{\text{fin}}^L(t)) \leq \ln \det(P_{\text{fin}}^U(t)) \leq \ln \det(P_{\text{fin}}(t)).
\]

Therefore, if the pair \((P_{\text{fin}}, Q_{\text{fin}}^{1/2})\) is stabilizable, \(\ln \det(P_{\text{fin}}^L(\infty))\) and \(\ln \det(P_{\text{fin}}^U(\infty))\) represent steady-state lower and upper bounds on the value of the cost function for the D-optimality criterion. Inequalities analogous to (36) hold with the cost functions (6) and (7) of the A- and E-optimality criteria.

We conclude this section with Prop. 2, which leveraging again the comparison theorem, studies the impact of the process noises \(w_i\) and \(n_i\) on the targets’ and vehicles’ position uncertainty (c.f. [26, Sect. VI]). In order to state the proposition, let us partition the matrix \(P_{\text{fin}}\) as

\[
P_{\text{fin}} = \begin{bmatrix} P_{TT} & P_{TV} \\ P_{VT} & P_{VV} \end{bmatrix},
\]

where \(P_{TT} \in \mathbb{R}^{3m \times 3m}\) is the covariance matrix corresponding to the targets’ position estimate, \(P_{VV} \in \mathbb{R}^{3n \times 3n}\) is the covariance matrix relative to the vehicles’ position estimate, and \(P_{TV} \in \mathbb{R}^{3m \times 3n}\) is the cross-correlation matrix between the targets and vehicles (obviously, \(P_{TV}^T = P_{VT}\)).

**Proposition 2 (Bounds on \(P_{TT}\) and \(P_{VV}\)):** Let \(I \subset \mathbb{R}_{\geq 0}\) be some interval, and \(t_0 \in I\). Let \(P_{TT}^L\) and \(P_{TT}^U\) be the solutions of the following RDEs on \(I\),

\[
\begin{aligned}
\dot{P}_{TT}^L &= F_{TT}^L P_{TT}^L + P_{TT}^L F_{TT}^{L^T} - P_{TT}^L C_{TT}^L(t) P_{TT}^L + \text{blkdiag}(Q_1, \ldots, Q_m, N_1, \ldots, N_1, \ldots, N_n), \\
\dot{P}_{TT}^U &= F_{TT}^U P_{TT}^U + P_{TT}^U F_{TT}^{U^T} - P_{TT}^U C_{TT}^U(t) P_{TT}^U + \text{blkdiag}(Q_1, \ldots, Q_m, N_1, \ldots, N_1, \ldots, N_n),
\end{aligned}
\]

(37)

with \(P_{TT}^L(t_0) = P_{TT}(t_0)\), and \(P_{TT}^U(t_0)\) symmetric. Then, if \(N_i' \geq N_i\) for a certain \(i \in \{1, \ldots, n\}\), we have that \(P_{TT}^L(t) \geq P_{TT}(t)\) for \(t \in I \cap [t_0, +\infty)\).

Analogously, let \(P_{TT}^U\) and \(P_{VV}\) be the solutions of the following RDEs on \(I\),

\[
\begin{aligned}
\dot{P}_{TV}^L &= F_{TV}^L P_{TV}^L + P_{TV}^L F_{TV}^{L^T} - P_{TV}^L C_{TV}^L(t) P_{TV}^L + \text{blkdiag}(Q_1, \ldots, Q_j', \ldots, Q_m, N_1, \ldots, N_1, \ldots, N_n), \\
\dot{P}_{TV}^U &= F_{TV}^U P_{TV}^U + P_{TV}^U F_{TV}^{U^T} - P_{TV}^U C_{TV}^U(t) P_{TV}^U + \text{blkdiag}(Q_1, \ldots, Q_j, \ldots, Q_m, N_1, \ldots, N_1, \ldots, N_n),
\end{aligned}
\]

with \(P_{TV}^L(t_0) = P_{TV}(t_0)\) and \(P_{TV}^U(t_0)\) symmetric. Then, if \(Q_j' \geq Q_j\) for a certain \(j \in \{1, \ldots, m\}\), we have that \(P_{TV}^U(t) \geq P_{TV}(t)\) for \(t \in I \cap [t_0, +\infty)\).

**Proof:** If \(N_i' \geq N_i\) for a certain \(i \in \{1, \ldots, n\}\), the application of Th. 1 to the RDEs in (37) leads to the condition,

\[
Y(t) \preceq \begin{bmatrix} P_{TV}(t) - P_{TT}(t) & \text{blkdiag}(P_{TV}(t) - P_{TT}(t)) \\
\text{blkdiag}(P_{TV}(t) - P_{TT}(t)) & P_{TV}(t) - P_{TV}(t) \end{bmatrix} \geq 0,
\]

(38)

for \(t \in I \cap [t_0, +\infty)\). From the property of the Schur complement of \(P_{TT}^U - P_{TT}\) with respect to \(Y\) [40, Prop. 8.2.4, ii)], (38) implies \(P_{TT}^U - P_{TT} \succeq 0\), i.e. \(P_{TT}^U \succeq P_{TT}\). The second part of the statement can be proved in an analogous way, by considering the Schur complement of \(P_{TV}^U - P_{TV}\) with respect to \(Y\). □

Prop. 2 states that an increase in the velocity uncertainty for one of the vehicles \(N_i' \geq N_i\), leads to an uncertainty growth in the targets’ position \(P_{TT}^U \succeq P_{TT}\). Viceversa, an increase in the velocity uncertainty for one of the targets \(Q_j' \geq Q_j\), directly translates into an uncertainty growth in the vehicles’ position \(P_{TV}^U \succeq P_{TV}\).

V. SIMULATION EXPERIMENTS

Extensive simulation experiments have been performed to illustrate the theory presented in the previous sections: in all our tests we used the cost function for the D-optimality criterion to design the gradient controller, because of its invariance under any nonsingular reparametrization [19, Sect. 6.1]. In the simulations, the initial velocity \(q_i(t_0) = q_i(0)\) of the aerial vehicles is set to zero.

Fig. 4(a) shows the trajectory of 4 aerial vehicles cooperatively tracking a target whose motion is governed by system (1), with \(F = -5 \text{diag}(10^{-3}, 10^{-4}, 10^{-2})\), \(G = I_3\), \(u_1(t) = [0, -10^{-2}, 0]^T\), and \(Q = 0.07 I_3\). The initial positions of the agents and target are, in meters, \(p_1(0) = [-3, 15, 0]^T\), \(p_2(0) = [3, 15, 0]^T\), \(p_3(0) = [0, 15, 5]^T\), \(p_4(0) = [0, 15, -5]^T\), \(x(0) = [0, 12, 0]^T\), and are respectively marked with circles and a square in Fig. 4(a). The parameters of the measurement model are \(a_0 = a_2 = 0.2\), \(a_1 = 3\), \(a_0 = \alpha_0 = 0.8\), and the gain and damping matrices are \(\Gamma = 0.5 I_3\) and \(B = 2 I_3\), respectively. Fig. 4(b) reports...
For consistency, in all the trials we used the same initialization of the time history of $x - \hat{x}_{\text{fus}}$; (d) Time history of $r_i, i \in \{1, \ldots, 4\}$ (colored) and $a_1 = 3 \text{ m}$ (black, dashed); (e) Mean and standard deviation of $\|x(T_{\text{ins}}) - \hat{x}_{\text{fus}}(T_{\text{ins}})\|$; $T_{\text{ins}} = 300 \text{ sec.}$, over 10 trials for $Q = \beta^2 \mathbf{I}_3, \beta \in \{0.01, 0.05, 0.1, 0.15, 0.2, \ldots, 0.4\}$.

The time history of $J = \ln \det(P_{\text{fus}})$ and Fig. 4(c) the target’s estimation error $x - \hat{x}_{\text{fus}}$. Fig. 4(d) displays the time evolution of $r_i, i \in \{1, \ldots, 4\}$ (colored): note that after about 150 seconds all the vehicles reach their sweet spot located at a distance $a_1 = 3 \text{ m}$ from the target (black, dashed line). It is also worth mentioning that although the “small-angle condition” is violated here (recall Remark 1), in this as well as in many other simulation experiments, we never experienced a significant performance degradation of our algorithm. Finally, Fig. 4(e) shows the influence of the process noise $w(t)$ on the steady-state target’s estimation error. In particular, in the figure we have reported the mean and standard deviation of $\|x(T_{\text{ins}}) - \hat{x}_{\text{fus}}(T_{\text{ins}})\|$; $T_{\text{ins}} = 300 \text{ sec.}$ is the final simulation time), over 10 trials for increasing process-noise power: $Q = \beta^2 \mathbf{I}_3, \beta \in \{0.01, 0.05, 0.1, 0.15, 0.2, \ldots, 0.4\}$.

For consistency, in all the trials we used the same initialization for the Kalman filter: $\hat{x}_{\text{fus}}(0) = x^T(0) + [1, -3, -2]^T, P_{\text{fus}}(0) = \mathbf{I}_3$.

Fig. 5(a) shows the trajectory of 4 aerial vehicles tracking a target and adopting the noncooperative estimation-and-control strategy described in Sect. II-D (for improving readability, the initial and final position of the agents is marked here with circles and black dots, respectively). The same parameters as those in Figs. 4(a)-(d) have been used in Fig. 5, the only difference being represented by the initial position of the vehicles and target, and by the gain and damping matrices, that are, respectively, $p_1(0) = [-5, 15, 0]^T, p_2(0) = [5, 15, 0]^T, p_3(0) = [0, 15, 5]^T, p_4(0) = [0, 15, -5]^T, x(0) = [0, 18, 0]^T$ (in meters), and $\mathbf{I} = 23 \mathbf{I}_3, \mathbf{B} = 15 \mathbf{I}_3$.

Fig. 5(b) reports the time history of the cost functions $J_i = \ln \det(P_i), i \in \{1, \ldots, 4\}$, and Fig. 5(c) the time evolution of the estimation errors $x - \hat{x}_i, i \in \{1, \ldots, 4\}$. (red agent 1, green agent 2, blue agent 3, magenta agent 4). In Fig. 5(b) we observe an agreement on the value of the cost functions $J_i, i \in \{1, \ldots, 4\}$ (corresponding to the existence of a feedback Nash equilibrium solution, c.f. Sect. II-D), and from a comparison of Fig. 5(b) and Fig. 4(b), we notice, as expected, that the individual estimates of the target’s position are poorer than the estimate obtained by fusing the measurements of the 4 agents.

Fig. 6(a) shows the trajectory of 2 targets and 3 vehicles performing ACLMT. In equations (25)-(26), we chose $F_1 = F_2 = -\text{diag}(10^{-3}, 10^{-3}, 10^{-4}), G_1 = G_2 = \mathbf{I}_3$, ...
bounds for matrix (black solid), and with the theoretical lower and upper lines associated with the actual position estimate’s covariance. The initial position of the target is marked with a square; (b) Time history of the cost functions $J_{1} = \ln \det(P_{1})$, $J_{2} = \ln \det(P_{2})$, $J_{3} = \ln \det(P_{3})$, and $J_{4} = \ln \det(P_{4})$ for all $t \geq 0$, as predicted by Prop. 2.

In Fig. 6(g), we compared the performance of our basic cooperative active target tracking algorithm (the positions of the vehicles are perfectly known here), with ACLMT, under equivalent conditions. The figure, in particular, shows the time history of the estimation error of a single target, obtained by four vehicles using the two approaches (black line and red line, respectively), with $\Gamma = 0.05 I_{3}$, $B = 50 I_{1}$ (for the former method), $a_{0} = a_{2} = 0.2$, $a_{1} = 5$, $a_{\phi} = a_{\phi} = 1$ and initial conditions, in meters, $p_{0}(0) = [−3, 18, 0]^{T}$, $p_{2}(0) = [−3, 10, −2]^{T}$, $p_{4}(0) = [6, 5, 1]^{T}$, $x_{0}(0) = [−5, 12, 1]^{T}$, $x_{2}(0) = [−5, 2, 4]^{T}$.

The parameters of the measurement model are $a_{0} = a_{2} = 0.2, 0.1 = 5, a_{0} = a_{0} = 1$ and the gain matrix $G = 0.09 I_{1}$.

For the sake of illustration, we chose the following simple association matrix,

$$A(x_{\text{tot}}(t)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \forall t \geq 0.$$

In this way, the first agent is assigned to the first target, the second agent to the second target, and the second agent to both targets. Fig. 6(b) shows the time history of $J = \ln \det(P)_{\text{fin}}$ (solid), $\ln \det(P)_{\text{fin}}$ (dotted) and $\ln \det(P)_{\text{fin}}$ (dashed): to generate this plot, we chose $\gamma_{m} = 12.5 m$. In Figs. 6(c)-(d), the estimation errors $\Delta_{x}(1:3) - \tilde{x}(1:3)$ and $\Delta_{x}(4:6) - \hat{x}(4:6)$ (red solid) of the first and second target, respectively, are compared against the ±3σ enveloping lines associated with the actual position estimate’s covariance matrix (black solid), and with the theoretical lower and upper bounds for $P_{\text{fin}}$ (dotted and dashed, respectively): similar position estimation errors were obtained for the three vehicles (not shown in Fig. 6).

Figs. 6(e)-(f) have been generated using the same parameters and initial conditions as those in Figs. 6(a)-(d), and they illustrate Prop. 2. Actually, Fig. 6(e) shows the time history of the eigenvalues of the matrix $P_{TT} - P_{TT}$ where $P_{TT}$ corresponds to $N_{1} = 1.05 I_{3}$ and $P_{TT}$ to $N_{2} = 0.03 I_{3}$ (note that there are only two distinct curves in Fig. 6(e), since some of the eigenvalues coalesce). Analogously, Fig. 6(f) reports the time history of the eigenvalues of the matrix $P_{VV} - P_{VV}$ in $\mathbb{R}^{9 \times 9}$, where $P_{VV}$ corresponds to $Q_{1} = 1.05 I_{3}$ and $P_{VV}$ to $Q_{1} = 0.05 I_{3}$. As it is evident from the two figures, the eigenvalues of $P_{TT} - P_{TT}$ and $P_{VV} - P_{VV}$ are nonnegative at all times, and hence $P_{TT}(t) \geq P_{TT}(t)$ and $P_{VV}(t) \geq P_{VV}(t)$ for all $t \geq 0$, as predicted by Prop. 2.

Finally, the Kalman filter was initialized with $\tilde{x}_{\text{fin}}(0) = [x_{1}^{T}(0), x_{2}^{T}(0), x_{3}^{T}(0), x_{4}^{T}(0), x_{5}^{T}(0), x_{6}^{T}(0), x_{7}^{T}(0), x_{8}^{T}(0), x_{9}^{T}(0)]$, $\hat{x}_{\text{fin}}(0) = [x_{1}^{T}(0), x_{2}^{T}(0), x_{3}^{T}(0), x_{4}^{T}(0), x_{5}^{T}(0), x_{6}^{T}(0), x_{7}^{T}(0), x_{8}^{T}(0), x_{9}^{T}(0)]$, $\tilde{P}_{\text{fin}}(0) = I_{15}$, respectively. From Fig. 6(g), we observe that although the transient performance of ACLMT is worse than that of the basic cooperative active target tracking algorithm, the steady-state target’s estimation errors of the two approaches are comparable.

Finally, Figs. 7(a)-7(c) illustrate the influence of the parameters $a_{0} = a_{0} \in \{0.05, 0.1, 0.25, 0.5, \ldots, 2\}$, $a_{0} \in \{0.01, 0.05, 0.25, 0.5, \ldots, 2\}$, and $\Delta_{1} = \Delta_{2} = \Delta_{3} = \gamma I_{15}$, $\gamma \in \{0.05, 0.075, 0.1, 0.25, 0.5, \ldots, 1\}$, on the bounds $\ln \det(P)_{\text{fin}}$ and $\ln \det(P)_{\text{fin}}$ (dotted and dashed, respectively) in ACLMT. Figs. 7(a)-7(c) were obtained using the same simulation parameters as those in Figs. 6(a)-(d) and, initializing the Kalman filter with $\tilde{x}_{\text{fin}}(0) = [x_{1}^{T}(0), x_{2}^{T}(0), x_{3}^{T}(0), x_{4}^{T}(0), x_{5}^{T}(0), x_{6}^{T}(0), x_{7}^{T}(0), x_{8}^{T}(0), x_{9}^{T}(0)]$, $\hat{x}_{\text{fin}}(0) = [x_{1}^{T}(0), x_{2}^{T}(0), x_{3}^{T}(0), x_{4}^{T}(0), x_{5}^{T}(0), x_{6}^{T}(0), x_{7}^{T}(0), x_{8}^{T}(0), x_{9}^{T}(0)]$, $\tilde{P}_{\text{fin}}(0) = I_{15}$. From the figures, it comes as no surprise that
the smaller $\alpha_\theta$, $\alpha_\phi$, $a_0$ and the “smaller” $\Delta_1$, $\Delta_2$, $\Delta_3$, the higher is the steady-state estimation accuracy of the target’s position.

**VI. CONCLUSIONS AND FUTURE WORK**

In this paper we have proposed an active target-tracking strategy to deploy a team of unmanned aerial vehicles along paths that minimize the uncertainty about the position of a
moving target. Both cooperative and noncooperative scenarios have been explored, and a new problem, called “Active Cooperative Localization and Multi-target Tracking” (ACLMT) has been introduced: in this problem, the aerial vehicles relocate themselves in the 3-D space in order to maximize both the accuracy of their own position estimate and that of multiple moving targets. For ACLMT, the monotonicity property of the Riccati differential equation arising from the Kalman-Bucy filter, which describes the propagation of the targets’/vehicles’ position uncertainty through time, allowed us to determine guaranteed performance bounds for our algorithm. The proposed theoretical results have been illustrated via extensive numerical simulations.

There is a series of interesting open issues that this paper has not tackled. First, in future research, we are going to extend the proposed results to heterogeneous teams of ground and aerial robots (e.g., modeled as more realistic nonholonomic vehicles, c.f. [41]), and to take advantage of the performance analysis of Sect. IV to synthesize more efficient controllers for ACLMT. We also aim to study the impact of team’s topology on the target(s) estimation accuracy in the case of limited inter-vehicle communication [25], and to test the proposed approach on real flying platforms (e.g. on commercial quadrotors [42]). Finally, in future works we will consider more sophisticated assignment algorithms for ACLMT in the case of exteroceptive sensors with limited range and/or aperture and cluttered environments [43]–[46], we will analyze the stability of the full closed-loop system in terms of the controller’s parameters (c.f. [10]), and we will investigate convex approximations to the cost functions (5)–(7), by exploiting, e.g., the concavity property of the Riccati differential equation of the Kalman-Bucy filter.

REFERENCES


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