On Connectivity Maintenance in Linear Cyclic Pursuit

Fabio Morbidi, Giulio Ripaccioli, Domenico Prattichizzo

Abstract—The paper studies the cyclic pursuit problem in presence of connectivity constraints among single-integrator agents. The robots, each one pursuing its leading neighbor along the line of sight rotated by a common offset angle, are supposed to have a communication set described by a disk of constant radius. Given the initial position of the agents, we determine the communication radii that preserve the connectivity of the robots while they rendezvous at a point or converge to an evenly spaced circle formation. The special case that the initial condition is a linear combination of the eigenvectors of the dynamic matrix of the system, is studied in detail. On the other hand, given the communication radii, we find the set of initial conditions that guarantee the robots remain always connected. As a final contribution, once assigned a “non-optimal” radius, we study the stability of the system describing the dynamics of the robotic network under variable connectivity levels.

I. INTRODUCTION

A. Motivation and related works

In the last few years we witnessed a growing interest in coordination and control of multi-agent systems [1]–[4]. The research in this area has been stimulated by the recent technological advances in wireless communications and processing units and by the observation that multiple agents can perform tasks more efficiently and reliably than a single robot. In this scenario, the cyclic pursuit problem has received increasing attention in the literature [5]–[8]. An underlying assumption in these works is that each robot can always communicate with its leading neighbor (from which it receives the position information). However, since real robots have limited connectivity range (usually modelled as a disk of finite radius centered on the robot), this does not always occur in practice [9]. Although the connectivity maintenance problem is well-known in the multi-agent literature, where several original solutions have been proposed (see, e.g., [10]–[13]), little attention has been devoted to the cyclic pursuit with connectivity constraints. In particular, a challenging but relatively unexplored problem to investigate is which is the effect of the initial position of the agents and of the value of the communication radius, on the connectivity of the robotic network.

B. Contributions

Following the general framework proposed in [6], this paper studies the connectivity maintenance problem in linear cyclic pursuit. The agents, each one pursuing its leading neighbor along the line of sight rotated by a common offset angle, are supposed to have a communication set modelled as a disk of constant radius.

The original contribution of the paper is twofold. First, given the initial position of the robots, we find the communication radii that preserve the connectivity of the robots while they rendezvous at a point or converge to an evenly spaced circle formation. The analytical expression of the minimum communication radius is determined when the initial position of the robots is a linear combination of the eigenvectors of the dynamic matrix of the system. In fact, in this special case the robots start moving from the vertices of a generalized regular polygon inscribed in the unit circle. By reversing the previous hypotheses, given the communication radii, we determine the set of initial conditions that ensure the robots remain always connected.

Second, once a “non-optimal” communication radius (i.e., a radius that does not guarantee the connectivity of the agent all the time) is assigned, we study the stability properties of the hybrid system describing the dynamics of the robots under variable connectivity levels.

C. Organization

Sect. II provides some basic definitions and results concerning the theory of circulant matrices and introduces the notation used through the paper. Sect. III reviews some of the main results in [6] and presents conditions on the communication radius and initial position of the agents to preserve the connectivity of the robotic network. In Sect. IV we study the stability of the system when the connectivity of the robots changes over time. In Sect. V, extensive simulation experiments illustrate the theory and show the effectiveness of the proposed designs. In Sect. VI, the main contributions of the paper are summarized and future research directions are highlighted.

II. MATHEMATICAL PRELIMINARIES AND NOTATION

A circulant matrix [5] of order $n$ is a square matrix of the form,

\[
C = \begin{bmatrix}
c_1 & c_2 & \ldots & c_n \\
c_n & c_1 & \ldots & c_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_2 & c_3 & \ldots & c_1
\end{bmatrix}
\]

The elements of each row of $C$ are identical to those of the previous row, but shifted one position to the right and wrapped around. The whole circulant matrix is thus determined by the first row and we can write $C = \text{circ}[c_1, c_2, \ldots, c_n]$. A circulant matrix is diagonalizable by the Fourier matrix, hence its eigenvalues and
eigenvectors can be readily determined. Let $A_1, A_2, \ldots, A_n$ be square matrices of order $m$. A block circulant matrix of type $(m, n)$ is a square matrix of order $mn$ of the form,

$$
\begin{bmatrix}
A_1 & A_2 & \cdots & A_n \\
A_n & A_1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
A_2 & A_3 & \cdots & A_1
\end{bmatrix}
$$

The whole matrix is determined by the first block row and can be written $\hat{A} = \text{circ}[A_1, A_2, \ldots, A_n]$. A block circulant matrix can be block diagonalized by means of the Fourier matrices, therefore the computation of the eigenvalues and eigenvectors is also in this case straightforward.

The following notation will be used throughout the paper: given a matrix $P \in \mathbb{R}^{n \times n}$, $P[j, j+1]$ denotes a $2 \times n$ matrix whose rows are the $j$-th and $(j+1)$-th row of $P$. blkdiag[$A_1, A_2, \ldots, A_n$] denotes a block diagonal matrix with square blocks $A_1, A_2, \ldots, A_n$ on the main diagonal. $0_n$ and $I_n$ indicate the $n \times n$ matrix of zeros and the identity matrix of order $n$, respectively.

### III. Conditions for Connectivity Maintenance

Consider $n$ mobile robots $R_i$, $i \in \{1, 2, \ldots, n\}$, in the plane and suppose that robot $i$ pursues the next $i+1$ modulo $n$ (henceforth, except where otherwise stated, all robot indices $i+1$ should be evaluated modulo $n$). Let $\xi_i(t) = [x_i(t), y_i(t)]^T, i \in \{1, 2, \ldots, n\}$, be the position of robot $i$ at time $t \geq 0$. The kinematics of each robot is described by a single integrator,

$$
\dot{\xi}_i(t) = u_i(t).
$$

Following the extension to the classic cyclic pursuit scenario proposed in [6], we suppose that each robot pursues the leading neighbor along the line of sight rotated of a common offset angle $\alpha \in [-\pi, \pi]$. The control input for each robot $i$ is thus given by

$$
u_i(t) = \kappa \mathbf{R}(\alpha) (\xi_{i+1}(t) - \xi_i(t)),
$$

where $\mathbf{R}(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ and $\kappa$ is a positive gain parameter that without loss of generality we will suppose equal to 1 in the remaining of the paper. The overall dynamics of the $n$ robots is,

$$
\dot{\xi}_i(t) = \hat{A} \xi_i(t),
$$

where $\xi = [\xi_1^T, \xi_2^T, \ldots, \xi_n^T]^T$ and $\hat{A}$ is a block circulant matrix:

$$
\hat{A} = \text{circ}[-\mathbf{R}(\alpha), \mathbf{R}(\alpha), 0_2, \ldots, 0_2].
$$

Note that at each instant of time, the multirobot system’s geometric configuration can be described by a pursuit graph [5] as follows.

**Definition 1 (Pursuit graph):** A pursuit graph consists of a pair $(V, E)$ such that:

1) $V$ is a finite set of vertices, $\text{card}(V) = n$, where each vertex $\xi_i = [x_i, y_i]^T, i \in \{1, 2, \ldots, n\}$, represents the position of the $i$-th robot in the plane;
2) $E$ is a finite set of directed edges, $\text{card}(E) = n$, where each edge $e_i : V \times V \rightarrow \mathbb{R}^2, i \in \{1, 2, \ldots, n\}$, is the vector from $\xi_i$ to its leading neighbor $\xi_{i+1}$.

In order to analyze the geometric patterns achievable by system (1) for different values of the offset angle $\alpha$, we need analytical expressions for the eigenvalues and eigenvectors of $\hat{A}$. The next two propositions have been proved in [6], while Prop. 3 is an original result of this paper.

**Proposition 1 (Eigenvalues of $\hat{A}$):** The eigenvalues of $\hat{A}$ are $\lambda^{\pm}_i = (\varphi_i - 1)e^{\pm jn}, i \in \{1, 2, \ldots, n\}$, where $\varphi_i \equiv 2(i-1)\pi/n$ and $j = \sqrt{-1}$.

It is easy to verify that $\lambda^{+}_1 = 0$ and that the following property holds true,

$$
\lambda^{+}_i = \lambda^{-}_{n-i+2}, \quad i \in \{2, 3, \ldots, n\},
$$

where $\bar{\lambda}$ denotes the complex conjugate of $\lambda$.

**Proposition 2 (Eigenvalues of $\hat{A}$):** The eigenvectors of $\hat{A}$ corresponding, respectively, to the eigenvalues $\lambda^{+}_1$ and $\lambda^{-}_1$ are

$$
w^{+}_i = [1, j, \varphi_i, j \varphi_i, \ldots, j \varphi_i^{-n+2}, j \varphi_i^{-n+1}]^T,
$$

$$
w^{-}_i = [1, -j, j \varphi_i, -j \varphi_i, \ldots, j \varphi_i^{-1}, -j \varphi_i^{-n+1}]^T.
$$

Note that $w^+_i$ and $w^-_i$ do not depend on the offset angle $\alpha$. Let $\mathbb{R}(w^+_i)$ and $\mathbb{S}(w^+_i)$ denote the real and imaginary part of $w^+_i$, $i \in \{2, 3, \ldots, n\}$.

**Proposition 3 (Orthogonality of the eigenvectors):** The vectors $\mathbb{R}(w^+_i)$ and $\mathbb{S}(w^+_i)$, (analogously, $\mathbb{R}(w^-_i)$ and $\mathbb{S}(w^-_i)$), $i \in \{2, 3, \ldots, n\}$, are orthogonal.

**Proof:** $\hat{A}$ is a normal matrix in fact

$$
\hat{A}^T \hat{A} = \text{circ}[-\mathbf{R}(\alpha), \mathbf{R}(\alpha), 0_2, \ldots, 0_2]^T \text{circ}[-\mathbf{R}(\alpha), \mathbf{R}(\alpha), 0_2, \ldots, 0_2] = \text{circ}[-\mathbf{R}(\alpha), \mathbf{R}(\alpha), 0_2, \ldots, 0_2] \text{circ}[-\mathbf{R}(\alpha), \mathbf{R}(\alpha), 0_2, \ldots, 0_2]^T = \hat{A}^T \hat{A}.
$$

In [6], the authors have studied all the geometric patterns formations achievable with $\alpha \in [-\pi, \pi]$ and showed that:

- For $|\alpha| < \pi/n$, the robots asymptotically converge to their initial center of mass (rendezvous at a point).
- For $|\alpha| = \pi/n$, the robots asymptotically converge to an evenly spaced circle formation, centered in their initial center of mass.
- For $|\alpha| \in (\pi/n, 2\pi/n)$, the robots asymptotically converge to an evenly spaced logarithm spiral formation, centered in their initial center of mass.

The initial center of mass of the robots is a fixed point for the system. It is a stable spiral for $|\alpha| < \pi/n$, a center for $\alpha = \pi/n$ and an unstable spiral for $|\alpha| \in (\pi/n, 2\pi/n)$. In this paper we will restrict to $|\alpha| \leq \pi/n$ and we will deal with the cyclic pursuit problem in presence of connectivity.
constraints among the robots. The communication set of robot $R_{i+1}$ is modeled as a disk of radius $r_{i+1}$ (see Fig. 1).

**Definition 2 (Connectivity among $R_{i+1}$ and $R_i$):**

$R_{i+1}$ is said to be connected with $R_i$ at time $t$ (to which it transmits the position information $\xi_{i+1}(t)$ necessary to compute the control input $u_i(t)$, if $R_i$ is inside its communication disk at time $t$. $R_{i+1}$ and $R_i$ are said to be connected for all $t \geq 0$, if $R_i$ keeps inside the communication disk of $R_{i+1}$ for all $t \geq 0$.

In Sect. III-A, we will determine the set of connectivity preserving radii when the initial condition $\xi(0)$ is assigned. In Sect. III-B, we study the special case that $\xi(0)$ is a linear combination of the eigenvectors of $A$. Finally, in Sect. III-C, we fix the radii $r_{i+1}$ and find the set of initial conditions that guarantee the robots remain always connected.

A. Connectivity preserving radii: general case

The following result is valid for general $\xi(0)$.

**Theorem 1 (Connectivity preserving radius):** Suppose that the communication disk of $R_{i+1}$, $i \in \{1, 2, \ldots, n\}$, has radius,

$$r_{i+1} \geq \max_{t \geq 0} \|\| T[2i-1, 2i] - T[2i+1, 2i+2] \|T^{-1}\| \xi(0)\|$$

where $T = [R(w_i^+), R(w_i^+), \ldots, R(w_i^+), R(w_i^+)]$ and $e^{\lambda (t)} = \text{blkdiag}(b_1(t), b_2(t), \ldots, b_n(t))$, with $b_i(t) = e^{R(\lambda^+ t)} R(\lambda^+ t)$. Then the robots achieve rendezvous or converge to a evenly spaced circle formation, without losing the connectivity for all $t \geq 0$.

**Proof:** The solution of system (1) has the form $\xi(t) = T e^{\lambda t} T^{-1} \xi(0)$, hence we can explicitly compute the distance among $R_{i+1}$ and $R_i$ at time $t$: $\|\| \xi_i(t) - \xi_{i+1}(t)\|\| = \|\| T[2i-1, 2i] - T[2i+1, 2i+2] \|T^{-1}\| \xi(0)\|$, $i \in \{1, 2, \ldots, n\}$. It is easy to verify that if $|\alpha| < \pi/n$, $\lim_{t \to \infty} \|\| \xi_i(t) - \xi_{i+1}(t)\|\| = 0$, while if $|\alpha| = \pi/n$, $\|\| \xi_i(t) - \xi_{i+1}(t)\|\| \leq c$ converges to a constant value.

By this definition, $\{\|\| \xi_i(t) - \xi_{i+1}(t)\|\|$ has always a global maximum that corresponds to the minimum value of $r_{i+1}$ sufficient to preserve the connectivity among $R_{i+1}$ and $R_i$ for all $t \geq 0$ (see Figs. 3(b), 3(d)).

Given the initial position $\xi(0)$ of the robots, Th. 1 provides us with the communication radii that preserve the connectivity among $R_{i+1}$ and $R_i$ while they rendezvous at a point or converge to a circle formation. Note that a condition equivalent to (3) can be derived by referring to $T_i^-, T_i^+$, instead to $T_i^+, T_i^-$.

**Remark 1 (Computation of the minimum radius $r_{i+1}$):**

Note that computing $r_{i+1} = \max_{t \geq 0} \|\| T[2i-1, 2i] - T[2i+1, 2i+2] \|T^{-1}\| \xi(0)\|$ analytically is not trivial in general and one should rely on numerical methods to find the maximum.

B. Connectivity preserving radii: special case

In order to find an analytical expression for the minimum connectivity preserving radius, we will now suppose that $\xi(0) \in \text{span}_R \{R(w_i^+), R(w_i^+), R(w_i^+), R(w_i^+), R(w_i^+), R(w_i^+), R(w_i^+), R(w_i^+), R(w_i^+)\}$, $k \in \{2, 3, \ldots, n\}$, (analogous results can be found for $\xi(0) \in \text{span}_R \{R(w_i^+), R(w_i^+)\}$, $k \in \{2, 3, \ldots, n\}$). Despite the apparent resemblance, note that $\text{span}_R \{R(w_i^+), R(w_i^+), R(w_i^+), R(w_i^+)\}$ is basically different from the eigenspace relative to the eigenvalue $\lambda_i^+$, $\text{span}_R \{R(w_i^+), R(w_i^+)\}$.

Let $\xi(0) = a R(w_i^+) + b R(w_i^+) + c R(w_i^+) + d R(w_i^+)$, $a, b, c, d \in R$: for $\alpha \in [-\pi, \pi]$, we have the following geometric pattern formations:

- If $k = 2$ and $-2\pi/n < \alpha < -\pi/n$, the robots move clockwise in an evenly spaced logarithm spiral formation centered in $(a, b)$.
- If $\alpha = -\pi/n$, the robots move clockwise in an evenly spaced circle formation centered in $(a, b)$.
- If $-\pi/n < \alpha < 2\pi/n$, the robots rendezvous at $(a, b)$.
- If $k = 2$ and $-2\pi/n < \alpha < \pi/n$, the robots move clockwise in an evenly spaced circle formation centered in $(a, b)$.
- If $\pi/n < \alpha < 2\pi/n$, the robots move counterclockwise in an evenly spaced logarithm spiral formation centered in $(a, b)$.

Note that if $\alpha = (1 - k)\pi/n + \pi/2$, $k \in \{2, 3, \ldots, n\}$, $\text{span}_R \{R(w_i^+)\}$ < 0 and the fixed point $(a, b)$ is a symmetric node.

As before, we will henceforth restrict to $\alpha$ corresponding to non positive $\text{span}_R \{R(w_i^+)\}$. The notion of generalized regular polygon [5] will be useful for the next derivations.

**Definition 3 (Generalized regular polygon (GRP)):**

Let $n$ and $\delta > n$ be positive integers so that $\beta n < n/\delta > 1$ is a rational number. Let $R$ be a positive rotation in the plane about the origin, through angle $2\pi/p$ and let $z_1 \neq 0$ be a point in the plane. Then, the points $z_{i+1} = R z_i$, $i \in \{1, 2, \ldots, n-1\}$ and edges $e_i = z_{i+1} - z_i$, $i \in \{1, 2, \ldots, n\}$, define a generalized regular polygon, which is denoted $\{\|\| \xi_i(t) - \xi_{i+1}(t)\|\|$.
polygon since its sides intersect at certain extraneous points, which are not included among the vertices. If \( n \) and \( \delta \) have a common factor \( m > 1 \), then \( \{p\} \) has \( n = n/m \) distinct vertices and \( \tilde{n} \) edges traversed \( m \) times. Before coming back to our problem, note that since the internal angle at every vertex of \( \{p\} \) is \( \pi(1 - 2\delta/n) \) [5], then the side of a polygon \( \{p\} \) inscribed in the unit circle has length \( 2\sin(\pi\delta/n) \).

**Lemma 1** (Eigenvectors of \( A \) and GRPs): Consider the vector \( \Re(w_k^+) \), (analogously \( \Im(w_k^+) \)), \( k \in \{2, 3, \ldots, n\} \) as a list of points in the plane. The pursuit graph defined by this sequence of points is a GRP \( \{n/\delta_k\} \) inscribed in the unit circle, where,

\[
\delta_k = \begin{cases} 
1 - k & \text{if } k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
(n - k - 1) & \text{otherwise.}
\end{cases}
\]

Let us illustrate Lemma 1 with a simple example:

**Example 1:** If \( n = 6 \),

\[
\begin{align*}
\Re(w_1^+) &= [1, 0, \frac{1}{2}, -\sqrt{3}, \frac{1}{2}, -\sqrt{3}, 1, 0, -\frac{1}{2}, \sqrt{3}, -\frac{1}{2}, \sqrt{3}]^T, \\
\Re(w_2^+) &= [1, 0, -1, 1, 0, 1, -1, 0, 1, 0, 1, 0]^T, \\
\Re(w_3^+) &= [1, 0, -1, 0, 1, 0, 1, 0, -1, 0, 1, 0]^T, \\
\Re(w_4^+) &= [1, 0, -1, 1, 0, 1, -1, 0, 1, 0, -1, 0]^T, \\
\Re(w_5^+) &= [1, 0, -1, 0, 1, 0, 1, 0, -1, 0, 1, 0]^T, \\
\Re(w_6^+) &= [1, 0, -1, 0, 1, 0, -1, 0, 1, 0, 1, 0]^T.
\end{align*}
\]

The pursuit graphs defined by \( \Re(w_k^+) \) and \( \Re(w_k^-) \) are ordinary regular hexagons (GRPs \{6/1\}) inscribed in the unit circle, with edges having opposite direction (see Fig. 2). Analogously, the pursuit graphs of \( \Im(w_k^+) \) and \( \Im(w_k^-) \) are GRPs \{6/2\} inscribed in the unit circle, with edges having opposite direction. Finally, the pursuit graph of \( \Re(w_k^+) \) is the GRP \{6/3\} with vertices \((1, 0)\) and \((-1, 0)\). Note that the the pursuit graph defined by \( \Im(w_k^+) \) is simply rotated of \( \pi/2 \) w.r.t. the graph of \( \Re(w_k^+) \), \( k \in \{2, \ldots, 6\} \), (cf. Prop. 3).

We are now ready to state the main result of this section.

**Theorem 2** (Connectivity preserving radii): Let \( \xi(0) = a\Re(w_1^+) + b\Im(w_1^+) + c\Re(w_2^+) + d\Im(w_2^+) \), \( a, b, c, d \in \mathbb{R} \), \( k \in \{2, 3, \ldots, n\} \). If the communication disk of the \( n \) robots has radius,

\[
r \geq 2\sin\left(\frac{\pi\delta_k}{n}\right) \left\| \frac{c}{d} \right\|,
\]

then the robots rendezvous at \((a, b)\) or move in an evenly spaced circle formation centered in \((a, b)\), without losing the connectivity for all \( t \geq 0 \).

**Proof:** The proof of the theorem is constructive. Let us define \( \nu = \sqrt{c^2 + d^2} \). Let us start with \( n = 2 \). Since

\[
\xi(0) = a\Re(w_1^+) + b\Im(w_1^+) + c\Re(w_2^+) + d\Im(w_2^+) \text{ then } ||\xi_2(t) - \xi_1(t)|| = 2\nu \|e^{\Re(t\lambda_k^+)t}\| \text{ and hence, by selecting a radius } r \geq 2\nu \text{, the robots will remain always connected.}
\]

Set \( n = 3 \). If we choose \( \xi(0) = a\Re(w_1^+) + b\Im(w_1^+) + c\Re(w_2^+) + d\Im(w_2^+) \), \( \xi(0) = a\Re(w_1^+) + b\Im(w_1^+) + c\Re(w_2^+) + d\Im(w_2^+) \), we have \( ||\xi_2(t) - \xi_1(t)|| = \sqrt{3}\nu \|e^{\Re(t\lambda_k^+)t}\|, ||\xi_2(t) - \xi_1(t)|| = \sqrt{3}\nu \|e^{\Re(t\lambda_k^+)t}\|, \) therefore if \( r \geq \sqrt{3}\nu \), the connectivity of the robots is preserved all the time. Set \( n = 4 \). If \( \xi(0) = a\Re(w_1^+) + b\Im(w_1^+) + c\Re(w_2^+) + d\Im(w_2^+) \) or \( \xi(0) = a\Re(w_1^+) + b\Im(w_1^+) + c\Re(w_2^+) + d\Im(w_2^+) \), we come up with the condition, \( r \geq \sqrt{2}\nu \), while if \( \xi(0) = a\Re(w_1^+) + b\Im(w_1^+) + c\Re(w_2^+) + d\Im(w_2^+) \), being \( \delta_3 = 2 \), we get, \( r \geq 2\nu \). By noticing that the constants before \( \nu \) in the previous inequalities are the length of the side of a GRP \( \{n/\delta_k\} \) inscribed in the unit circle, we obtain condition (4).

**Remark 2** (Formation symmetry): It is easy to show that \( \Re(w_1^+) \), \( \Re(w_2^+) \), \( \Re(w_3^+) \), \( \Re(w_4^+) \), \( k \in \{2, 3, \ldots, n\} \) is an \( A \)-invariant subspace. This means that if \( \xi(0) \in \Re(w_1^+) \), \( \Re(w_2^+) \), \( \Re(w_3^+) \), \( \Re(w_4^+) \), then \( \xi(t) \) will remain in there \( \forall t > 0 \). In particular, it can be proved that the cyclic group formation symmetry \( C_q \), where \( q \) is a divisor of \( n \), is preserved at each time instant [8].

**Corollary 1** (Convergence speed): Let \( |\alpha| < \pi/n \). If \( \delta_j > \delta_j, h, j \in \{2, 3, \ldots, n\} \), then \( \Re(\lambda_k^+) < \Re(\lambda_j^+) \).

Loosely speaking, this means that the bigger \( \delta_k \) the faster the robots converge to the point \((a, b)\).

**C. Connectivity preserving initial conditions**

By reversing the hypotheses of Th. 1, let us now suppose that the radii of the communication disks are assigned and that we want to find the set of initial conditions that guarantee the robots remain connected \( \forall t \geq 0 \). The next theorem proposes a simple solution to this problem.

**Theorem 3** (Connectivity preserving initial conditions): Let the radii \( r_{i+1}, i \in \{1, 2, \ldots, n\} \) of the communication disks, be given. If the initial condition \( \xi(0) \) satisfies,

\[
||\xi(0)|| \leq \min\{s_1, s_2, \ldots, s_n\},
\]

where \( s_{i+1} = r_{i+1}(\max_{t \geq 0} ||(T[2i - 1, 2i] - T[2i + 1, 2i + 2])e^{\lambda^+(T^{-1})t}||)^{-1} \), then the robots achieve rendezvous or converge to a evenly spaced circle formation, without losing the connectivity for all \( t \geq 0 \).

**Proof:** We have that, \( \max_{t \geq 0} ||(T[2i - 1, 2i] - T[2i + 1, 2i + 2])e^{\lambda^+(T^{-1})t}|| \leq \max_{t \geq 0} ||(T[2i - 1, 2i] - T[2i + 1, 2i + 2])e^{\lambda^+(T^{-1})t}|| \xi(0) ||. \) If we impose that \( \max_{t \geq 0} ||(T[2i - 1, 2i] - T[2i + 1, 2i + 2])e^{\lambda^+(T^{-1})t}|| \xi(0) || \leq
to show the existence of a common quadratic Lyapunov proposition is omitted. To prove the statement is sufficient if the communication radii of all the robots, we obtain condition (5).

Note that if the communication radii are all equal to \( r \), from (4) a feasible set of initial conditions is simply given by
\[
\alpha \Re(w_k^+) + b \Im(w_k^+) + c \Re(w_k^-) + d \Im(w_k^-), \quad k \in \{2, 3, \ldots, n\},
\]
where \( a, b \) can assume any value and \( c, d \) are such that
\[
\sqrt{c^2 + d^2} \leq r(2\sin(\pi \delta_k/n))^{-1}.
\]

IV. STABILITY UNDER VARIABLE CONNECTIVITY LEVELS

In this section we will suppose that \( \xi(0) \) is a general vector of \( \mathbb{R}^{2n} \) and that the communication radius \( r_{i+1} \) of robot \( R_{i+1} \) does not satisfy condition (3). This means that the connectivity among \( R_{i+1} \) and \( R_i \) will be lost during a certain time interval. Since \( R_i \) does not receive the position information \( \xi_{i+1} \) from \( R_{i+1} \) in this interval, a natural choice here is to set the control input \( u_i(t) = -R(\alpha) \xi_i(t) \). The overall system dynamics is driven in this case by a block triangular matrix \( \hat{A} \) whose eigenvalues \( \lambda^\pm = -e^{\pm j\pi/4} \) have algebraic multiplicity \( n \). Hence, if \( |\alpha| < \pi/2 \) the system is asymptotically stable. Note that \( \hat{A} \) still has a block triangular structure (with same eigenvalues) when multiple robots simultaneously lose the connectivity. Our dynamic system basically operates in 2\( n \) modes for \( |\alpha| \leq \pi/n, n > 2 \):

- Modes \( 0 \cdots 2^{n-2} \): At least one robot has lost the connectivity, matrix \( \hat{A} \) is block triangular.
- Mode \( 2^{n-1} \): All the robots are connected, matrix \( \hat{A} \) is block circulant.

For the stability analysis it is convenient to introduce the following hybrid system:

\[
\dot{\xi}(t) = A_{\sigma(t)} \xi(t),
\]

where

\[
A_{\sigma(t)} = \begin{bmatrix}
-R(\alpha) & \gamma_1(t) R(\alpha) & 0_2 & 0_2 & \cdots & 0_2 \\
0_2 & -R(\alpha) & \gamma_2(t) R(\alpha) & 0_2 & \cdots & 0_2 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
\gamma_n(t) R(\alpha) & 0_2 & 0_2 & 0_2 & \cdots & -R(\alpha)
\end{bmatrix}
\]

with

\[
\gamma_i(t) = \begin{cases}
0 & \text{if } \|\xi_i(t) - \xi_{i+1}(t)\| > \tau_{i+1} \\
1 & \text{else}
\end{cases},
\]

and \( \sigma(t) = \mathbb{D}(\gamma(t)) \), i.e., \( \sigma \) is the decimal coding of the binary vector \( \gamma \). In particular, note that \( A_{2^{n-1}} = \text{circ}[-R(\alpha), R(\alpha), 0_2, \ldots, 0_2] \) (all the robots are connected) and that \( A_0 = \text{blkdiag}[-R(\alpha), -R(\alpha), \ldots, -R(\alpha)] \) (all the robots have lost the connectivity). The proof of the next proposition is omitted. To prove the statement is sufficient to show the existence of a common quadratic Lyapunov function for system (6) (see, e.g., [14]).

**Proposition 4** (Stability with variable connectivity): If \( |\alpha| \leq \pi/n, n > 2 \), the robots rendezvous at the point \( \xi_{ss} \) or converge to an evenly spaced circle formation centered in \( \xi_{ss} \), where \( \xi_{ss} \) depends only on \( \xi(0) \) and \( \tau_{i+1} \).

Note that when at least one robot loses the connectivity, the agents start to converge towards the origin. Hence, there will be a time instant when all the agents will eventually recover the connectivity and then start to converge to a point or to a circle formation.

V. SIMULATION EXPERIMENTS

Simulation experiments have been conducted to illustrate the theory and show the effectiveness of the proposed designs. Figs. 3(a), 3(c) show the trajectory of 4 robots for \( \alpha = \pi/6 \) and \( \alpha = \pi/4 \). The initial conditions are \( \xi(0) = [2.8, 3.5, 3.5, 4, 2.5, 5.5, -1, 5]^T \) and \( \xi(0) = [-2, 4, 2, 4, 3, 5, -2, -2]^T \), respectively. Figs. 3(b), 3(d) report the time history of \( \|\xi_i(t) - \xi_{i+1}(t)\|, i \in \{1, \ldots, 4\} \) and the minimum connectivity preserving radii \( r_{i+1}^* \).

In Figs. 3(e)-3(g), \( \xi(0) = \Re(w_1^+) + \Im(w_1^+) + \Re(w_2^+) + 3 \Im(w_2^+) = [2, 4, 4, 0, 0, -2, -2, 2]^T \). Figs. 3(e), 3(f) report the trajectory of the robots for \( \alpha = \pi/10 \) and \( \alpha = \pi/4 \). At \( t = 0 \), the robots are on the vertices of a square centered in \( (1, 1) \) with side \( \sqrt{20} \) (the length of the minimum connectivity preserving radius). Fig. 3(g) shows that for \( \alpha = \pi/4 \), the distance among the robots decreases as \( e^{-\sqrt{2}t} \).

In Fig. 4, \( \alpha = \pi/6 \) and the radius of the communication disks is \( \tau_{i+1} = \tau = 2.3, i \in \{1, \ldots, 4\} \). \( \xi(0) \) is the same as in Fig. 3(a). Fig. 4(c) reports the sequence of operating modes of the system: three matrices, \( A_{12}, A_{14} \) and \( A_{15} \) are involved in this sequence. In the interval \([0, t_{s1}]\), \( R_1 \) is not connected with \( R_4 \) and \( R_3 \) is not connected with \( R_3 \). At time \( t_{s1} \), \( R_4 \) recovers the connectivity while \( R_3 \) remains unconnected until time \( t_{s2} \). For \( t > t_{s2} \) all the robots are connected and asymptotically converge to the point \( \left( \frac{1}{4} \sum_{i=1}^{4} x_i(t_{s2}), \frac{1}{4} \sum_{i=1}^{4} y_i(t_{s2}) \right) \).

VI. CONCLUSIONS AND FUTURE WORK

The paper studies the connectivity maintenance problem in linear cyclic pursuit. The agents, each one pursuing its leading neighbor along the line of sight rotated by a common offset angle, are supposed to have a communication set described by a disk of constant radius. Given the initial position of the robots, we determine the communication radii that preserve the connectivity of the robots while they rendezvous at a point or converge to evenly spaced circle formation. On the other hand, given the communication radii we find the set of all initial conditions that guarantee the robots remain always connected. Finally, once fixed a “non-optimal” radius, we study the stability of the hybrid system that describes how the connectivity of the robotic network changes over time. The extension of our results to unicycle robots is a subject of future research. Future investigations will also address the problem of enforce connectivity via a distributed control action.

REFERENCES

**Fig. 3.** General $\xi(0)$: Trajectory of the robots for (a) $\alpha = \pi/6$, (c) $\alpha = \pi/4$; (b)-(d) Time history of $\|\xi_i(t) - \xi_{i+1}(t)\|$ and minimum connectivity preserving radii $r^*_i$. **Special $\xi(0)$:** Trajectory of the robots for (e) $\alpha = \pi/10$, (f) $\alpha = \pi/4$; (g) Time history of $\|\xi_i(t) - \xi_{i+1}(t)\|$ for $\alpha = \pi/4$.

**Fig. 4.** Variable connectivity: (a) Trajectory of the robots for $\alpha = \pi/6$ and $\tau = 2.3$; (b) Time history of $\|\xi_i(t) - \xi_{i+1}(t)\|$; (c) Sequence of modes.


