Leader-Follower Formation Control and Visibility Maintenance of Nonholonomic Mobile Robots

Fabio Morbidi

Advisor: Prof. D. Prattichizzo
Co-Advisor: Prof. A. Vicino

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Leader-Follower Formation Control and Visibility
Maintenance of Nonholonomic Mobile Robots

Fabio Morbidi,

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mORBIdI@dii.unisi.it.
3.3.3 A geometrical interpretation of the EOJ singularity ..... 39
3.3.4 Necessity of the observability condition ..... 39
3.3.5 Observer design ..... 40
3.4 Input-state feedback control ..... 41
3.5 Simulations and experimental results ..... 42
  3.5.1 Simulations ..... 42
  3.5.2 Experiments ..... 44
3.6 Dealing with distant robots ..... 48
3.7 Observer via Immersion and Invariance ..... 50
  3.7.1 Observer design via I&I: an overview ..... 51
  3.7.2 I&I range estimator ..... 53
  3.7.3 Formation control and closed-loop stability ..... 56
  3.7.4 Simulation results ..... 59
3.8 Conclusions and future work ..... 59

4 Visibility maintenance via controlled invariance ..... 63
  4.1 Introduction ..... 63
  4.2 The linear constrained regulation problem ..... 65
  4.3 The visibility maintenance problem ..... 71
    4.3.1 Extension: rejection of UBB disturbances ..... 76
    4.3.2 Extension: VMP on a circle ..... 78
    4.3.3 Extension: chain of \( n \) robots ..... 79
  4.4 Simulation results ..... 81
  4.5 Conclusions and future work ..... 83

5 Conclusions and future research ..... 85
  5.1 Summary of contributions ..... 85
  5.2 Future research directions ..... 87

Bibliography ..... 89

Index ..... 97
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I’m very fond of the following excerpt – drawn from L. Schwartz’s biography – that I believe perfectly describes the daily effort that every researcher makes to pursue the Truth:

“A researcher must accept to slog away at a problem for a hour, a day, or all his life. Rather, he uses up his energy excessively with respect to the results, he asks himself several questions, he gropes in the dark, he moves forward step by step. This is a hard task; then, at a certain point, the illumination comes. It is often unexpected but it is the result of a huge amount of unsuccessful reflections.”

Siena
June 19, 2009
Symbols and abbreviations

≜ equal by definition
A, X sets or polyhedra or manifolds
∂X the boundary of the set X
a, x scalars or vectors
∅ the empty set
N the set of all natural numbers
Z the set of all integer numbers
R the set of all real numbers
N^n the set of all n-tuples of natural numbers
R^n the set of all n-tuples of real numbers
R^{m\times n} the set of all m \times n matrices with elements in R
[t_0, t_1] a closed interval
C^1([t_0, t_1], \mathbb{R}^n) the space of continuously differentiable n-valued functions on [t_0, t_1]
L^\infty the space of bounded functions of time
\text{sign } x the signum function: \forall x > 0, \text{ sign } x = 1; \forall x < 0, \text{ sign } x = -1
\langle x, y \rangle the inner or scalar product of vectors x and y
\|x\|_2 the Euclidean norm of vector x: \|x\|_2 \triangleq \sqrt{\langle x, x \rangle}
|x| the absolute value of the real number x
A, X matrices or linear transformations
I_n the n \times n identity matrix
A^T the transpose of A
A^{-1} the inverse of A (A square nonsingular)
det A the determinant of A
\text{end of discussion}
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>AUV</td>
<td>Autonomous underwater vehicle</td>
</tr>
<tr>
<td>UAV</td>
<td>Unmanned aerial vehicle</td>
</tr>
<tr>
<td>EKF</td>
<td>Extended Kalman filter</td>
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<tr>
<td>EOJ</td>
<td>Extended Output Jacobian</td>
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<tr>
<td>NEES</td>
<td>Normalized estimation error squared</td>
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<tr>
<td>VMP</td>
<td>Visibility maintenance problem</td>
</tr>
<tr>
<td>UUB</td>
<td>Uniformly ultimate bounded</td>
</tr>
<tr>
<td>LCRP</td>
<td>Linear constrained regulation problem</td>
</tr>
<tr>
<td>DLCRP</td>
<td>Linear constrained regulation problem with additive disturbances</td>
</tr>
<tr>
<td>UBB</td>
<td>Unknown but bounded</td>
</tr>
<tr>
<td>I&amp;I</td>
<td>Immersion and invariance</td>
</tr>
<tr>
<td>FME</td>
<td>Fourier-Motzkin elimination</td>
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1.1 Motivation and related work

Multiagent systems and cooperative control are nowadays becoming research topics of increasing popularity, especially within the robotics and control community (Olfati-Saber et al. 2007, Murray 2007, Bullo et al. 2009). The research in this area has been stimulated by the recent technological advances in wireless communications and processing units, and by the observation that multiple agents can perform tasks far beyond the capabilities of a single robot (Kumar et al. 2008). This trend has been also supported by the effort to mimic biological systems, such as, e.g., flocks of birds, schools of fish, swarms of insects (see Fig. 1.1), where self-organizing or emergent behaviors result from agents that appear to act autonomously (Reynolds 1987, Strogatz 2003, Olfati-Saber 2006).

Potential applications of multiagent systems are multifarious (see Fig. 1.2) and include terrain and utilities inspection, disaster monitoring, environmental surveillance

Figure 1.1: Examples of biological systems exhibiting cooperative behaviors: (a) flock of geese, (b) school of fish, (c) swarm of bees, (d) team of ants.
1. Introduction

Figure 1.2: Working scenarios of multiagent systems: (a) fire and environmental monitoring, (b) ocean sampling (Slocum glider®, image courtesy of Teledyne Webb Research), (c) search and rescue, (d) traffic control (image courtesy of Microdrones GmbH), (e) manipulation and assembling (image courtesy of Interaction Lab, University of Southern California), (f) military espionage (image courtesy of US Department of Defence), (g) automatic warehouse management (image courtesy of Kiva Systems), (h) planetary exploration (rover Spirit, image courtesy of NASA Jet Propulsion Laboratory, 2004), (i) soccer playing (RoboCup, Humanoid League, 2005).

(Casbeer et al. 2006, Susca et al. 2008) and modeling (Lynch et al. 2008), ocean sampling (Leonard et al. 2007), traffic control, search and rescue, structures moving and assembling (Pereira et al. 2004), warehouse management, mine hunting (Pavone and Frazzoli 2007), military espionage and reconnaissance, mapping of unknown and hazardous environments, planetary exploration, soccer playing (Parker 2008).

Research in multiagent systems did not restrict to wheeled robots (Das et al. 2002) (see Fig. 1.3), but dealt with different vehicle typologies as well, such as, e.g. autonomous underwater vehicles (AUVs) (Fossen 1994, Edwards et al. 2004), unmanned aerial vehicles (UAVs) (Beard and McLain 2003, Flint et al. 2002), microsatellites (Schaub
1.1. Motivation and related work

Figure 1.3: Team of wheeled mobile robots (image courtesy of GRASP lab, University of Pennsylvania).

et al. 2000, Burns et al. 2000), microrobots (Seyfried et al. 2005). Last frontiers of research are going towards larger and larger groups (from hundreds to thousands) of small, cheap agents, so that the loss of some of them during a mission has not a critical effect on the accomplishment of a common task. This trend has been confirmed by the recent announcement of the maiden flight of a micro-aerial vehicle of the size of housefly (Wood 2008), and of the creation of swarms of these insect-like robots in a near future.


The focus of this thesis is on the formation control and visibility maintenance problems, that will be described in the remaining of this section.

For its wide range of applicability, the formation control problem has stimulated a great deal of research in recent years. By formation control we simply mean the problem of controlling the relative position and orientation of the robots in a group while allowing the group to move as a whole. Existing approaches to robot formation control generally fall into three categories: behavior based, virtual structure and leader following.
1. Introduction

Figure 1.4: Formation composed of a leader (L) and \( n \) followers \((F_1, \ldots, F_n)\). The desired formation in this illustrative example (see also Chapter 2), is defined by the parameters \((d_i, \phi_i)\), \( i = 1, 2, \ldots, n \).

In the behavior based approach (Matarić 1992, Balch and Arkin 1998, Lawton et al. 2003, Antonelli et al. 2008) several desired behaviors (e.g. collision avoidance, formation keeping, target seeking) are prescribed to each robot. Robot final action is derived by weighting the relative importance of each behavior. The theoretical formalization and mathematical analysis of this approach is difficult and consequently it is not easy to guarantee the convergence of the formation to a desired configuration.

The virtual structure approach (Tan and Lewis 1997, Ren and Beard 2004) considers the robot formation as a single virtual rigid structure so that the behavior of the robotic system is assimilable to that of a physical object. Desired trajectories are not assigned to each single robot but to the entire formation as a whole. In this case the behavior of the robot formation is predictable and consequently the stability analysis can be easily performed. Nevertheless a large inter-robot communication bandwidth is required.

In the leader-follower approach (see Fig. 1.4) a robot of the formation, designed as the leader, moves along a predefined trajectory while the other robots, the followers, are to maintain a desired posture (distance and orientation) to the leader (Das et al. 2002, Tanner et al. 2004, Léchevin et al. 2006, Gustavi and Hu 2008). The main criticism to leader following is that the formation does not tolerate leader faults and exhibits poor disturbance rejection properties. In spite of these deficiencies the leader-follower approach is particularly appreciated in the literature because of its simplicity and scalability.
1.1. Motivation and related work

To gain insight into the notion of leader following, two simple examples are reported in Fig. 1.5. Five mobile robots (a leader and four followers) achieve a desired wedge formation, starting from an initial arbitrary pose: in Fig. 1.5(a) the leader moves along a rectilinear trajectory and in Fig. 1.5(b) along a circular path.

So far, we have not made any specific assumption on the sensing capabilities of the robots. In a realistic scenario, it is reasonable to assume that the agents have some sensory limitations. For example, it is apparent that two robots can communicate only if the first one keeps inside a disk region (representing the approximate extent of the electromagnetic field used for data exchange), centered on the second (see Fig. 1.6(a)). Analogously, a robot can sense only those agents that lie within a circular sector, representing the footprint of a range sensor with limited angular visibility (e.g. a laser range finder), (see Fig. 1.6(b)). With this in mind, it is evident that if multiple robots have to perform complex tasks, it may be very hard to guarantee they can communicate or sense each other all the time. These are typically referred to as the connectivity and visibility maintenance problems in the literature: to solve them means to find suitable (possibly distributed) control laws for the agents, so that the connectivity/visibility is actively enforced at each time.

The formation control and visibility maintenance problems become particularly challenging when the vehicles can not freely move in all directions but have some kinematic
1. Introduction

Figure 1.6: Sensory limitations: (a) Robot $R_1$ can communicate only with those robots that are within a disk of radius $r$; (b) $R_1$ can sense only those robots that lie within a circular sector.

Constraints, that typically impose a zero lateral velocity for the robot. Vehicles incorporating these constraints are called nonholonomic and have been the subject of extensive research in the last decades (see e.g., (Kolmanovsky and McClamroch 1995)). Two simple nonholonomic vehicles, the unicycle and the Dubins vehicle (Dubins 1957, Reeds and Shepp 1990), are commonly found in the literature and used to model wheeled robots and aerial vehicles.

Leader following of nonholonomic mobile robots with input constraints is a common thread of this thesis, as it is explained in the next section.

1.2 Contributions and organization of the thesis

The first part of the thesis deals with leader-follower formation control of unicycle robots from two different but complementary points of view.

In the first scenario (Chapter 2), we suppose that the robots are assigned given constraints on the linear velocity and trajectory curvature, and study which is the effect of these bounds on the admissible trajectories of the leader that guides the formation. A new geometric approach to the asymptotic stabilization of the robots to a desired formation is presented. A peculiar feature that distinguishes our control strategy from others existing in the literature (that impose stricter constraints on the structure of the formation), is that the follower position is not rigidly fixed with respect to the leader but varies in suitable circle arcs centered in the leader reference frame.
A simplified setup composed of a leader-follower pair of robots is first considered and some preliminary results established. These results are subsequently extended to a general class of multirobot formations, called hierarchical formations, characterized by elementary leader-follower units interconnected as the nodes of a rooted tree graph. In this framework, we provide recursive formulae for the maximum velocity and curvature allowed to the main leader, such that the robots asymptotically achieve the desired formation while respecting their input constraints.

In a second and more application-oriented scenario (Chapter 3), the unicycle robots are supposed to be equipped only with panoramic cameras (see Fig. 1.7) and the localization problem for the leader-follower system is addressed by using a new observability condition valid for general nonlinear systems and based on the Extended Output Jacobian. As an improvement over the existing literature, we do not assume to know any camera calibration parameter (mirror shape or focal length), nor the pose of any stationary landmark: only the view-angle to the other robots is provided by each camera, but not the distance, that is estimated via a nonlinear observer (the extended Kalman filter (EKF)). An input-state feedback control law is designed to stabilize the robots to a desired formation. As a second contribution, thanks to our observability condition we identify those trajectories of the leader that preserve system observability. An insightful geometrical interpretation of vision-based formation localizability is also proposed. As an extension, we show how the proposed feedback control scheme can be modified to address the case of distant robots. Finally, a globally convergent reduced-order observer based on the Immersion and Invariance (I&I) technique is proposed as an alternative to the EKF, and the closed-loop stability is proved in this case via Lyapunov arguments.
The final part of the thesis (Chapter 4) deals with the visibility maintenance problem (VMP) for a leader-follower pair of Dubins-like vehicles with bounded control inputs. Differently from the literature, where potential field methods are typically applied, we address the VMP by using the notion of controlled invariance. The key idea is to interpret the nonlinear model describing the relative dynamics of the leader and the follower as a linear system with model parameter uncertainty, with the velocity input of the leader playing the role of an external disturbance. The VMP can then be reformulated as linear constrained regulation problem with additive disturbances (DLCRP). New positive $\mathcal{D}$-invariance conditions for general linear uncertain systems with parametric disturbance matrix are introduced and used to solve the VMP when box bounds on the visibility set, control inputs and disturbances are considered. Analytical conditions for the solution of the VMP are obtained by symbolically solving the set of linear inequalities defining the polytope of all the feasible state feedback matrices, using the Fourier-Motzkin elimination method. The proposed design procedure can be easily adapted to provide the control with UBB disturbances rejection capabilities, and extended to the case of robots moving on a circle and to chains of $n$ robots.

In Chapter 5 the main contributions of the thesis are summarized and possible avenues of future research are highlighted.
Chapter 2

Formation control with input constraints

Madness vs. Intelligence

“Men have called me mad; but the question is not yet settled, whether madness is or is not the loftiest intelligence – whether much that is glorious – whether all that is profound – does not spring from disease of thought – from moods of mind exalted at the expense of the general intellect.”

from “Eleonora” – E.A. Poe

Abstract

This chapter presents a new geometric approach to the stabilization of a leader-follower formation of unicycle robots with input constraints. A peculiar feature of the proposed control strategy is that the follower position is not rigidly fixed with respect to the leader but varies in suitable circle arcs centered in the leader reference frame. A simplified setup composed of a leader-follower pair of robots is first considered and some preliminary results established. These results are subsequently extended to a general class of multirobot formations, called hierarchical formations, and recursive formulae for the maximum velocity and curvature allowed to the main leader such that the robots asymptotically achieve the desired formation while respecting their input constraints, are provided.

The material of this section is based on (Consolini et al. 2006, Consolini et al. 2007a, Morbidi and Prattichizzo 2007, Morbidi et al. 2007, Consolini et al. 2007b, Consolini et al. 2008, Consolini et al. 2007, Under review).

2.1 Introduction

In this chapter we will deal with leader-follower formations of unicycle robots and investigate the effect of input constraints on the possible trajectories of the leader and admissible positions of the followers with respect to the leader. A new geometric approach to the asymptotic stabilization of the robots to a desired formation is proposed. A peculiar feature that distinguishes our control strategy from others existing in the literature (that impose stricter constraints on the structure of the formation
Formation control with input constraints

Figure 2.1: Structure of a simple hierarchical formation. The main leader $R_0$ guides the formation. One of the 7 leader-follower units is highlighted: $R_1$ is the relative leader of $R_4$. Robots $R_5$, $R_6$ and $R_7$ act only as followers in the formation.

(see e.g. (Das et al. 2002)), is that the follower position is not rigidly fixed with respect to the leader but varies in suitable circle arcs centered in the leader reference frame.

In the first part of the chapter, a simplified setup composed of a leader-follower pair of robots is considered and some preliminary results are presented. These results are subsequently extended to hierarchical formations. Hierarchical formations are characterized by elementary leader-follower units interconnected as the nodes of a rooted tree graph (see Fig. 2.1). The main leader $R_0$ drives the formation while all the other robots $R_i$, $i = 1, 2, \ldots, n$ act both as followers and leaders (with the exception of the leaves of the tree, that are only followers). In this framework, we provide recursive formulae for the maximum velocity and curvature allowed to the main leader, such that the robots asymptotically achieve the desired formation while respecting their input constraints. Note that the term “hierarchical” is motivated here by the rooted tree structure of the formation, which introduces a partial order on the nodes, given by the distance of each node from the common root.

Further results, including a reformulation of the leader-follower formation control problem as a disturbance decoupling problem and an extension to robots with more involved kinematics, are briefly discussed at the end of the chapter. Simulation experiments illustrate the theory and show the effectiveness of the proposed designs.

The rest of the chapter is organized as follows. In Sect. 2.2 some basic definitions are provided and the problem studied in the chapter is formulated. In Sects. 2.3 and 2.4 the new geometric approach to the stabilization of a two-robot and a multirobot formation is proposed. In Sect. 2.5 simulation experiments illustrate the theory and show the
2.2. Basic definitions and problem formulation

Effectiveness of the proposed designs. In Sect. 2.6 some variations on the main theme are discussed. In Sect. 2.7 the major contributions of the chapter are summarized and possible lines of future research are suggested.

The following notation will be used through the chapter: \( \forall a, b \in \mathbb{R}, a \land b = \min\{a, b\}, \ a \lor b = \max\{a, b\}; \ \forall \theta \in \mathbb{R}, \tau(\theta) = (\cos \theta, \sin \theta)^T, \nu(\theta) = (-\sin \theta, \cos \theta)^T; \ S^1 \) denotes the quotient space \( \mathbb{R}/\mathcal{R} \) equipped with the canonical topology, \( \mathcal{R} \) being the equivalence relation \( x \sim y \Leftrightarrow x - y = 2m\pi, \ m \in \mathbb{Z} \).

2.2 Basic definitions and problem formulation

Consider the following definition of robot as a velocity controlled unicycle model.

2.1. DEFINITION. \( \mathbf{R} = (x, y, \theta)^T \in C^1([0, +\infty), \mathbb{R}^3) \) is called a unicycle robot with initial condition \( \overline{R} \in \mathbb{R}^3 \) and control \((v, \omega)^T \in C^0([0, +\infty), \mathbb{R}^2)\), if the following system is verified,

\[
\begin{align*}
\dot{x} &= v \cos \theta \\
\dot{y} &= v \sin \theta \\
\dot{\theta} &= \omega \\
(x(0), y(0), \theta(0))^T &= \overline{R}.
\end{align*}
\]

System (2.1) represents the differential kinematics of the unicycle robot and \( \forall t \geq 0 \) such that \( v(t) \neq 0, \kappa(t) = \omega(t)/v(t) \) is the curvature of the path followed by the robot at time \( t \).

Denote by \( P(t) = (x(t), y(t))^T \) the position of \( \mathbf{R} \) at time \( t \), \( \theta(t) \) its heading, \( \tau(\theta(t)) \) the normalized velocity vector and \( \nu(\theta(t)) \) the normalized vector orthogonal to \( \tau(\theta(t)) \). Hence, \( \{\tau(\theta(t)), \nu(\theta(t))\} \) represents the robot reference frame at time \( t \) (see Fig. 2.2).
2.2. Definition. The triple \( (D, \Phi, \mathcal{N}) \), where \( D = (d_1, d_2, \ldots, d_n)^T \in \mathbb{R}^n \), \( \Phi = (\phi_1, \phi_2, \ldots, \phi_n)^T \in \mathbb{R}^n \) and \( \mathcal{N} = (l_1, l_2, \ldots, l_n)^T \in \mathbb{N}^n \), is said to be admissible (to make a formation) if \( \forall i = 1, \ldots, n, d_i > 0, |\phi_i| < \frac{\pi}{2}, l_i \in \{0, \ldots, i-1\} \).

The following definition introduces the notion of hierarchical formation (or \( (D, \Phi, \mathcal{N}) \)-formation) used in the sequel (see Fig. 2.3(a)).

2.3. Definition. Let \( (D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n \) be an admissible triple. We say that \( n+1 \) robots \( R_0, R_1, \ldots, R_n \) are in \( (D, \Phi, \mathcal{N}) \)-formation with leader \( R_0 \) at time \( t \), if \( \forall i = 1, \ldots, n \),

\[
P_i(t) = P_i(t) + d_i \tau(\theta_i(t) + \phi_i),
\]

(2.2)

and simply that \( R_0, R_1, \ldots, R_n \) are in \( (D, \Phi, \mathcal{N}) \)-formation with leader \( R_0 \), if (2.2) holds \( \forall i = 1, \ldots, n, \forall t \geq 0 \).

\( R_0 \) guides the formation and is called the main leader. \( R_i \) is the relative leader to the \( i \)-th follower \( R_i \) and each pair \( (R_0, R_i) \) is referred to as a leader-follower unit. Definition 2.3 states that \( n+1 \) robots \( R_0, R_1, \ldots, R_n \) are in \( (D, \Phi, \mathcal{N}) \)-formation with leader \( R_0 \) if for any robot \( R_i \), the position \( P_i \) of the follower \( R_i \) is always at distance \( d_i \) from the position \( P_i \) of the relative leader \( R_i \) and the angle between vectors \( \tau(\theta_i) \) and \( P_i - P_i \) is constantly equal to \( \phi_i \), i.e. the position \( P_i \) of the relative leader remains fixed with respect to the reference frame \( \{\tau(\theta_i), \nu(\theta_i)\} \).

Note that usually in the literature, the follower’s position is kept fixed with respect to the frame of the leader (see, e.g. (Das et al. 2002, Morbidi, Mariottini and Prattichizzo 2008)). On the contrary, in the proposed setup it is the relative leader’s position that is kept fixed in each follower’s reference frame.

An equivalent way to state Definition 2.3 is the following. Set the error vector,

\[
E_i(t) = P_i(t) - (P_i(t) + d_i \tau(\theta_i(t) + \phi_i)),
\]

then \( R_0, R_1, \ldots, R_n \) are in \( (D, \Phi, \mathcal{N}) \)-formation with leader \( R_0 \) if and only if \( E_i(t) = 0, \forall i = 1, \ldots, n, \forall t \geq 0 \).

It is worth noting that hierarchical formations represent rooted trees in the context of graph theory. In the special case that \( l_i = 0 \) for all \( i \), all the robots have \( R_0 \) as relative leader, while if \( l_i = i - 1 \) for all \( i \), a “convoy-like” formation is obtained, i.e. a formation where the \( i \)-th robot has the \( (i-1) \)-th as relative leader. Finally, remark that if \( n = 1 \), then \( D = d_1, \Phi = \phi_1, \mathcal{N} = 0 \), that is \( R_0 \) and \( R_1 \) are in \( (d_1, \phi_1, 0) \)-formation with leader \( R_0 \), which is the case considered in (Consolini et al. 2008), (see Fig. 2.3(b)).

2.4. Definition. Let \( (D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n \) be an admissible triple. We say that \( n+1 \) robots \( R_0, R_1, \ldots, R_n \) are asymptotically in \( (D, \Phi, \mathcal{N}) \)-formation with leader \( R_0 \), if \( \forall i = 1, \ldots, n \),

\[
\lim_{t \to \infty} E_i(t) = P_i(t) - (P_i(t) + d_i \tau(\theta_i(t) + \phi_i)) = 0.
\]
2.3. Stabilization of a two-robot formation

Before dealing with Problem 2.1, it is worth studying a simpler case, namely the asymptotic stabilization of a two-robot formation. To understand the choice of the controls we are going to use, consider the following remark.

Figure 2.3: (a) A sample \((D, \Phi, \mathcal{N})\)-formation with \(D = (d_1, d_2, d_3)^T\), \(\Phi = (\phi_1, \phi_2, \phi_3)^T\) and \(\mathcal{N} = (0, 0, 2)^T\); (b) \((d_1, \phi_1, 0)\)-formation.

The following problem will be studied in next sections.

2.1. Problem. Let \((D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n\) be an admissible triple. Let \(R_0, R_1, \ldots, R_n\) be \(n + 1\) robots and \(V_i, K_i^-, K_i^+, i = 0, \ldots, n\) be real constants. Find the velocity and curvature bounds \(V_0, K_0^-, K_0^+\) of robot \(R_0\) that guarantee the existence of controls \(v_i, \omega_i, i = 1, \ldots, n\) for robot \(R_i\) such that for any initial condition \(R_i\), the robots \(R_0, R_1, \ldots, R_n\) are asymptotically in \((D, \Phi, \mathcal{N})\)-formation with leader \(R_0\) and the following constraints are satisfied: \(\forall i = 1, \ldots, n, \forall t \geq 0,\)

\[
0 < v_i(t) \leq V_i, \quad K_i^+ \leq \kappa_i(t) \leq K_i^-.
\]

Roughly speaking, given the curvature and velocity bounds of each follower, we want to find the bounds on the velocity and curvature of the main leader such that there exist controls for the followers, for which the robots asymptotically achieve the desired formation while respecting their input constraints.

2.3 Stabilization of a two-robot formation

Before dealing with Problem 2.1, it is worth studying a simpler case, namely the asymptotic stabilization of a two-robot formation. To understand the choice of the controls we are going to use, consider the following remark.
2.1. REMARK. Suppose that two robots $\mathbf{R}_0$ and $\mathbf{R}_1$ are in $(d_1, \phi_1, 0)$-formation with leader $\mathbf{R}_0$. Then it must be,

$$v_1(t) = v_0(t) \frac{\cos(\beta_1(t) - \phi_1)}{\cos \phi_1}, \quad \omega_1(t) = v_0(t) \frac{\sin \beta_1(t)}{d_1 \cos \phi_1},$$

(2.3)

where $\beta_1(t) \triangleq \theta_0(t) - \theta_1(t)$.

In fact, since

$$E_1(t) = P_0(t) - (P_1(t) + d_1 \tau(\theta_1(t) + \phi_1)) = 0,$$

differentiating it, we get that,

$$\dot{P}_0(t) = \dot{P}_1(t) + d_1 \dot{\theta}_1(t) \nu(\theta_1(t) + \phi_1),$$

that is

$$v_0(t) \tau(\theta_0(t)) = v_1(t) \tau(\theta_1(t)) + d_1 \omega_1(t) \nu(\theta_1(t) + \phi_1),$$

which implies, multiplying by the rotation matrix $\mathcal{R}(-\theta_1(t)) \triangleq (\tau(-\theta_1(t)), \nu(-\theta_1(t)))$, that

$$v_0(t) \tau(\beta_1(t)) = v_1(t)(1, 0)^T + d_1 \omega_1(t) \nu(\phi_1).$$

Therefore it has to be,

$$v_1 = v_0 \left[ \cos \beta_1 + \frac{\sin \beta_1 \sin \phi_1}{\cos \phi_1} \right] = \frac{v_0 \cos(\beta_1 - \phi_1)}{\cos \phi_1},$$

(2.3)

$$\omega_1 = v_0 \frac{\sin \beta_1}{d_1 \cos \phi_1}.$$

The stabilization control strategy we are going to present consists of two steps. In the first step $\mathbf{R}_1$ moves with maximum linear and angular velocities until its direction is sufficiently close to that of $\mathbf{R}_0$. In the second step, $\mathbf{R}_1$ performs the control defined in Remark 2.1 with an added stabilizing term in order to reduce the error asymptotically to zero. Note that since $\forall \mathbf{a}, \mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$ such that $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle \neq 0$,

$$\mathbf{a} = \frac{\langle \mathbf{a}, \mathbf{a}_2 \rangle}{\langle \mathbf{a}_1, \mathbf{a}_2 \rangle} \mathbf{a}_1 + \frac{\langle \mathbf{a}, \mathbf{a}_1 \rangle}{\langle \mathbf{a}_2, \mathbf{a}_1 \rangle} \mathbf{a}_2,$$

then the error vector $E_1$ can be decomposed with respect to vectors $\tau(\theta_1)$ and $\nu(\theta_1 + \phi_1)$ as follows,

$$E_1 = \frac{\langle E_1, \nu(\theta_1 + \phi_1) \rangle}{\langle \tau(\theta_1), \nu(\theta_1 + \phi_1) \rangle} \tau(\theta_1) + \frac{\langle E_1, \tau(\theta_1) \rangle}{\nu(\theta_1 + \phi_1), \tau(\theta_1 + \phi_1)} \nu(\theta_1 + \phi_1) = E_{1\tau} \tau(\theta_1) + E_{1\nu} \nu(\theta_1 + \phi_1),$$

(2.4)
2.3. Stabilization of a two-robot formation

where (see Fig. 2.4),

\[
E_{1\tau} = \frac{\langle E_1, \tau(\theta_1 + \phi_1) \rangle}{\cos \phi_1}, \quad E_{1\nu} = \frac{\langle E_1, \nu(\theta_1) \rangle}{\cos \phi_1}.
\]

Let \( \Gamma_1^1 = \{ \gamma \in S^1 | (K_0^- - \epsilon) d_1 \cos \phi_1 \leq \sin \gamma \leq (K_0^+ + \epsilon) d_1 \cos \phi_1 \} \). The stabilizing controller that will be referred to in Proposition 2.1 is given by,

\[
v_1(t) = \begin{cases} 
V_1, & \text{if } \beta_1(t) \notin \Gamma_1^1 \\
\frac{v_0(t) \cos(\beta_1(t) - \phi_1) + \eta_1(t) \langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle}{\cos \phi_1}, & \text{if } \beta_1(t) \in \Gamma_1^1
\end{cases} \quad (2.5)
\]

\[
\omega_1(t) = \begin{cases} 
\frac{V_1 K_1^+}{d_1 \cos \phi_1}, & \text{if } \beta_1(t) \notin \Gamma_1^1 \text{ and } K_1^+ \geq 0 \\
\frac{V_1 K_1^-}{d_1 \cos \phi_1}, & \text{if } \beta_1(t) \notin \Gamma_1^1 \text{ and } K_1^+ < 0 \\
\frac{v_0(t) \sin \beta_1(t) + \eta_1(t) \langle E_1(t), \nu(\theta_1(t)) \rangle}{d_1 \cos \phi_1}, & \text{if } \beta_1(t) \in \Gamma_1^1
\end{cases} \quad (2.6)
\]

which adds correcting terms proportional to the error components to the controls \( v_1, \omega_1 \) in (2.3), (see the proof of Proposition 2.1 for the definition of the proportionality factor \( \eta_1(t) \)).

To understand the genesis of some of the constants and constraints which appear in the statement of Proposition 2.1, consider the situation presented in Fig. 2.5 where the robots \( R_0 \) and \( R_1 \) are in \((d_1, \phi_1, 0)\)-formation and are following two circles \( C_0 \) and \( C_1 \) of radius \( r_0, r_1 \) and curvature \( K_0 > 0, K_1 > 0 \), respectively. If \( d_1 > 0, |\phi_1| < \frac{\pi}{2} \) and \( K_0 < \frac{1}{d_1 \cos \phi_1} \), the following relations hold true:

\[
r_1 = d_1 \sin \phi_1 + \sqrt{r_0^2 - d_1^2 \cos^2 \phi_1}.
\]
2. Formation control with input constraints

\[ K_1 = \frac{1}{d_1 \sin \phi_1 + \sqrt{\frac{1}{K_0} - d_1^2 \cos^2 \phi_1}}. \]  

(2.7)

Indeed, as shown in Proposition 2.1, the value of $K_1$ given in (2.7) is precisely the maximum positive curvature of a follower robot whose leader is following a path of maximum positive curvature less than $K_0$. An analogous reasoning can be made for negative curvatures. The following definition is introduced to provide concise statements in Proposition 2.1.

2.5. DEFINITION. Let $d > 0$ and $\phi : |\phi| < \frac{\pi}{2}$. Define the following strictly increasing monotone functions: $\forall \kappa$ such that $|\kappa| \leq \frac{1}{d \cos \phi}$, set,

\[ \chi_{d,\phi}(\kappa) = \frac{1}{d \sin \phi + (\text{sign} \kappa) \sqrt{\frac{1}{\kappa^2} - d^2 \cos^2 \phi}}, \]

and $\forall \kappa \in \mathbb{R}$, set,

\[ \chi_{d,\phi}^{-1}(\kappa) = \frac{\text{sign} \kappa}{\sqrt{d^2 \cos^2 \phi + (\frac{1}{\kappa} - d \sin \phi)^2}}, \]

with the convention that $\chi_{d,\phi}(0) = \chi_{d,\phi}^{-1}(0) = 0$.

Therefore in the example of Fig. 2.5, $K_1 = \chi_{d_1,\phi_1}(K_0)$ (see equ. (2.7)) and $K_0 = \chi_{d_1,\phi_1}^{-1}(K_1)$. Let $d_1 > 0$, $\phi_1 : |\phi_1| < \frac{\pi}{2}$ and let $W_0, V_0, K^{-}_0, K^{+}_0, V_1, K^{-}_1, K^{+}_1$ be suitable constants such that,

\[
\begin{align*}
0 < W_0, V_0, &V_1, K^{-}_1 \leq K^{+}_1, \\
\text{if } \phi_1 < 0, \quad &-\frac{1}{d_1 \cos \phi_1} \quad < K^{-}_0 \leq K^{+}_0 < \frac{1}{d_1 \cos \phi_1}, \quad \text{if } \phi_1 > 0 \\
\text{if } \phi_1 \geq 0, \quad &-\frac{1}{d_1} \quad < K^{-}_0 \leq K^{+}_0 < \frac{1}{d_1} \quad \text{if } \phi_1 \leq 0.
\end{align*}
\]

(2.8)
2.3. Stabilization of a two-robot formation

Set \( \tilde{K}_0^\pm = \chi_d^{-1}_1(K_1^\pm) \) and,
\[
\tilde{V}_0 = V_1 \cos \phi_1 \left( \cos(0 \wedge (\arcsin(K_0^+ d_1 \cos \phi_1) - \phi_1) \wedge (\phi_1 - \arcsin(K_0^- d_1 \cos \phi_1))) \right)^{-1}.
\]

2.1. PROPOSITION. In the previous notation and hypotheses, let \( R_0 \) be a robot satisfying the following constraints for all \( t \geq 0 \),
\[
0 < W_0 \leq v_0(t) \leq V_0, \quad K_0^- \leq \kappa_0(t) \leq K_0^+.
\]
If the following conditions are satisfied:
\[
\tilde{K}_0^- < K_0^- \leq K_0^+ < \tilde{K}_0^+, \quad V_0 < \tilde{V}_0,
\]
then for any initial condition \( \mathbf{R}_1 \) of the robot \( \mathbf{R}_1 \), the controls \( v_1, \omega_1 \) given by (2.5) and (2.6) are such that,
\[
\lim_{t \to \infty} E_1(t) = 0,
\]
and the following constraints are verified for robot \( \mathbf{R}_1 \),
\[
0 < W_1 \leq v_1(t) \leq V_1, \quad K_1^- \leq \kappa_1(t) \leq K_1^+, \quad \forall t \geq 0,
\]
where,
\[
W_1 = \frac{W_0}{2 \cos \phi_1} \cos \left( (\arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1) - \phi_1) \vee (\phi_1 - \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1)) \right),
\]
and \( \epsilon > 0 \) is a sufficiently small constant. Furthermore \( \forall \epsilon > 0, \exists \bar{t} \geq 0 : \forall t \geq \bar{t}, \)
\[
\arcsin((K_0^- - \epsilon) d_1 \cos \phi_1) \leq \theta_0(t) - \theta_1(t) \leq \arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1).
\]

Proof: Take any \( \epsilon > 0 \) sufficiently small such that,
\[
\begin{align*}
&\begin{cases} 
\text{if} \quad \phi_1 < 0, & -\frac{1}{d_1 \cos \phi_1} \\
\text{if} \quad \phi_1 \geq 0, & -\frac{1}{d_1}
\end{cases} < K_0^- - \epsilon \leq K_0^+ + \epsilon < \begin{cases} 
\frac{1}{d_1 \cos \phi_1}, & \text{if} \quad \phi_1 > 0 \\
\frac{1}{d_1}, & \text{if} \quad \phi_1 \leq 0
\end{cases} \\
\tilde{K}_0^- < K_0^- - \epsilon \leq K_0^+ + \epsilon < \tilde{K}_0^+,
\end{align*}
\]
\[
V_0 \cos(0 \wedge (\arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1) - \phi_1) \wedge (\phi_1 - \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1))) < V_1 \cos \phi_1.
\]

Let the controls be given by (2.5) and (2.6), where,
\[
\eta_1(t) = \frac{(v_0(t) - W_0/2) \cos(\beta_1(t) - \phi_1)}{|\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \wedge \frac{(K_0^+ + \epsilon/2 - \kappa_0(t)) \wedge (\kappa_0(t) - (K_0^- - \epsilon/2))d_1 \cos \phi_1}{|\langle E_1(t), \nu(\theta_1(t)) \rangle|} \wedge V_1 \cos \phi_1 \frac{v_0(t) \cos(\beta_1(t) - \phi_1)}{|\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \wedge \frac{v_0(t)(K_1^+ d_1 \cos(\beta_1(t) - \phi_1) - \sin \beta_1(t))}{|\langle E_1(t), \nu(\theta_1(t)) \rangle| + |K_1^+||\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \wedge \frac{v_0(t)(\sin \beta_1(t) - K_1^- d_1 \cos(\beta_1(t) - \phi_1))}{|\langle E_1(t), \nu(\theta_1(t)) \rangle| + |K_1^-||\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \wedge M_1,
\]
\[
(2.16)
\]
with the convention that $\frac{1}{0} = +\infty$ and $M_1$ is a positive gain constant. Note that the complexity of $\eta_1(t)$ is due to the necessity of satisfying both the curvature and velocity constraints. First of all we remark that,

$$\exists \bar{t} \geq 0 : \beta_1(\bar{t}) \in \Gamma^1_\epsilon,$$  \hspace{1cm} (2.17)

In fact, suppose for instance that $K_1^+ \geq 0$, then if $\forall t \geq 0, \beta_1(t) \notin \Gamma^1_\epsilon$, by (2.10),

$$\dot{\beta}_1(t) = \omega_0(t) - \omega_1(t) = \kappa_0(t) v_0(t) - K_1^+ V_1 \leq K_0^+ v_0(t) - K_1^+ V_1$$

$$\leq (K_0^+ v_0(t) - K_1^+ V_1) - \epsilon v_0(t) \leq (K_0^+ v_0(t) - K_1^+ V_1) - \epsilon W_0 \leq -\epsilon W_0,$$

being $(\dot{K}_0^+ V_0 - K_1^+ V_1) \leq 0$. In fact, suppose for simplicity that,

$$\cos(0 \wedge (\arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1) - \phi_1)) \wedge (\phi_1 - \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1))) = \cos 0 = 1,$$

by (2.12) and (2.15),

$$\dot{K}_0^+ V_0 - K_1^+ V_1 \leq \frac{V_1 \cos \phi_1}{\sqrt{d_1^2 \cos^2 \phi_1 + ((K_1^+)^{-1} - d_1 \sin \phi_1)^2}} - K_1^+ V_1$$

$$\leq \frac{V_1 (\cos \phi_1 - \sqrt{\cos^2 \phi_1 + (K_1^+ d_1 - \sin \phi_1)^2})}{\sqrt{d_1^2 \cos^2 \phi_1 + ((K_1^+)^{-1} - d_1 \sin \phi_1)^2}} \leq 0.$$

Therefore,

$$\beta_1(t) \leq -\epsilon W_0 t + \beta_1(t_0), \ \forall t \geq 0,$$

which implies straightaway property (2.17). Set $\beta_1^{-1}(\Gamma^1_\epsilon) = \{ t \geq 0 \mid \beta_1(t) \in \Gamma^1_\epsilon \}$, then by (2.6), $\forall t \in \beta_1^{-1}(\Gamma^1_\epsilon)$:

$$\dot{\beta}_1(t) = \omega_0(t) - \omega_1(t) = \omega_0(t) - \frac{v_0(t) \sin \beta_1(t) + \eta_1(t) \langle E_1(t), \nu(\theta_1(t)) \rangle}{d_1 \cos \phi_1}$$

$$= \frac{v_0(t) (\kappa_0(t) d_1 \cos \phi_1 - \sin \beta_1(t)) - \eta_1(t) \langle E_1(t), \nu(\theta_1(t)) \rangle}{d_1 \cos \phi_1},$$

which implies by definition (2.16) of $\eta_1(t)$ that if,

$$\beta_1(\bar{t}) = \arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1), \ (\beta_1(\bar{t}) = \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1)),$$

then,

$$\dot{\beta}_1(\bar{t}) \leq -\epsilon V_0/2, \ (\dot{\beta}_1(\bar{t}) \geq \epsilon W_0/2).$$

This implies, together with (2.17), that,

$$\exists \bar{t} \geq 0 : \forall \bar{t} \in [0, \bar{t}), \ \beta_1(t) \notin \Gamma^1_\epsilon \text{ and } \forall t \geq \bar{t}, \ \beta_1(t) \in \Gamma^1_\epsilon,$$  \hspace{1cm} (2.18)
which gives (2.14). To prove that constraints (2.12) are verified for \( v_1 \), remark that by (2.5) and (2.18), \( v_1(t) = V_1, \forall t \in [0, \bar{t}] \) and \( \forall t \geq \bar{t}, \)

\[
v_1(t) = \frac{v_0(t) \cos(\beta_1(t) - \phi_1) + \eta_1(t)\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle}{\cos \phi_1}.
\]

Therefore by definition of \( \eta_1(t) \) and (2.9),

\[
v_1(t) \geq \frac{v_0(t) \cos(\beta_1(t) - \phi_1) - \eta_1(t)\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle}{\cos \phi_1} \geq \frac{W_0}{2} \frac{\cos(\beta_1(t) - \phi_1)}{\cos \phi_1}.
\]

Therefore \( v_1(t) > 0, \forall t \geq 0 \), by the following property:

\[
\exists c_1 > 0 : \cos(\beta_1(t) - \phi_1) \geq c_1, \forall t \geq \bar{t}.
\]

(2.19)

In fact, suppose for instance that \( \phi_1 > 0 \), by (2.14) and (8.8)

\[
\beta_1(t) - \phi_1 \leq \arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1) - \phi_1 < \arcsin(1) - \phi_1 = \frac{\pi}{2} - \phi_1,
\]

and

\[
\beta_1(t) - \phi_1 \geq \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1) - \phi_1 > - \arcsin(\cos \phi_1) - \phi_1 = -\frac{\pi}{2},
\]

which implies (2.19) and (2.13). Furthermore by (2.5) and (2.16), \( v_1(t) \leq V_1, \forall t \geq 0 \).

To verify constraints (2.12) for \( \kappa_1(t) \), remark first of all that they are verified by (2.5) and (2.6) if \( t \in [0, \bar{t}] \). If \( t \geq \bar{t} \), inasmuch as \( \eta_1(t) > 0 \) (see equ. (2.20)):

\[
\kappa_1(t) = \frac{\omega_1(t)}{v_1(t)} = \frac{v_0(t) \sin \beta_1(t) + \eta_1(t)\langle E_1(t), \nu(\theta_1(t)) \rangle}{d_1 v_0(t) \cos(\beta_1(t) - \phi_1) + \eta_1(t)\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle}
\]

\[
\leq \frac{v_0(t) \sin \beta_1(t) + \eta_1(t)\langle E_1(t), \nu(\theta_1(t)) \rangle}{d_1 v_0(t) \cos(\beta_1(t) - \phi_1) - \eta_1(t)\langle \text{sign} K_1^+, \langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle \rangle} \leq K_1^+,
\]

since, by (2.16),

\[
\eta_1(t) \leq \frac{v_0(t)(K_1^+ d_1 \cos(\beta_1(t) - \phi_1) - \sin \beta_1(t))}{\|\langle E_1(t), \nu(\theta_1(t)) \rangle\| + |K_1^+\|\|\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle\|},
\]

and analogously, \( \kappa_1(t) \geq K_1^-, \forall t \geq \bar{t} \); therefore constraints (2.12) are completely satisfied. To conclude the proof it remains to verify (2.11). Differentiating the error \( E_1(t) \), by (2.5), (2.6) and recalling (2.4), we get that \( \forall t \geq \bar{t} : \)

\[
\dot{E}_1(t) = v_0(t) \tau(\theta_0(t)) - v_0(t) \left( \frac{\cos(\beta_1(t) - \phi_1)}{\cos \phi_1} \tau(\theta_1(t)) + d_1 \frac{\sin \beta_1(t)}{\cos \phi_1} \nu(\theta_1(t) + \phi_1) \right)
\]

\[
- \eta_1(t) \left( \frac{\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle}{\cos \phi_1} \tau(\theta_1(t)) + \frac{\langle E_1(t), \nu(\theta_1(t)) \rangle}{\cos \phi_1} \nu(\theta_1(t) + \phi_1) \right) = -\eta_1(t) E_1(t).
\]
Therefore,
\[
\frac{d}{dt} \|E_1(t)\|^2 = -2 \eta_1(t) \|E_1(t)\|^2, \quad \forall \ t \geq \bar{t},
\]
and then (2.11) is satisfied since the following property holds true,
\[
\exists c > 0 : \eta_1(t) \geq \left( \frac{c}{\|E_1(t)\|} \land M_1 \right), \quad \forall \ t \geq \bar{t}.
\]  \tag{2.20}

In fact to verify (2.20), first of all remark that by (2.9) it follows straightaway that,
\[
\left( K_0^+ + \frac{\epsilon}{2} - \kappa_0(t) \right) \land \left( \kappa_0(t) - (K_0^- - \frac{\epsilon}{2}) \right) \geq \frac{\epsilon}{2}, \quad \forall \ t \geq 0.
\]
Moreover by (2.15),
\[
V_1 \cos \phi_1 - v_0(t) \cos(\beta_1(t) - \phi_1) \geq V_1 \cos \phi_1 - \nabla_0 \cos(0 \land (\arcsin(K_0^+ d_1 \cos \phi_1) - \phi_1) \land (\phi_1 - \arcsin(K_0^- d_1 \cos \phi_1))) = c_2 > 0,
\]
and in addition by (2.14), \( \forall \ t \geq \bar{t} \),
\[
K_1^+ - \frac{\sin \beta_1(t)}{d_1 \cos(\beta_1(t) - \phi_1)} \geq K_0^+ - \frac{\text{sign} (K_0^+ + \epsilon)}{\sqrt{K_0^+} - d_1 \cos \phi_1 + d_1 \sin \phi_1} > 0,
\]
since \((K_0^+ + \epsilon) < \bar{K}_0^+\), which implies that,
\[
\exists c_3^+ > 0 : K_1^+ d_1 \cos(\beta_1(t) - \phi_1) - \sin \beta_1(t) \geq c_3^+.
\]
Analogously,
\[
\exists c_3^- > 0 : \sin \beta_1(t) - K_1^- d_1 \cos(\beta_1(t) - \phi_1) \geq c_3^-.
\]

Therefore bringing together (2.19) with the previous inequalities, we obtain (2.20) by definition of \( \eta_1(t) \). \[ \square \]

Proposition 2.1 states that given a follower robot \( R_1 \) with velocity and curvature constrained by (2.12), if the leader \( R_0 \) maneuvers with velocity and curvature sufficiently bounded, then the control law given by (2.5) and (2.6) allows \( R_1 \) to asymptotically achieve the desired formation for every initial state. The bounds on the leader's velocity and curvature (see conditions (2.8) and (2.10)) depend on the formation’s parameters \((d_1, \phi_1)\) and on the follower’s capability of maneuver.

Note that condition (2.14) states that the difference \( \theta_0(t) - \theta_1(t) \) between the robots’ headings remains bounded during the motion. The following remark gives a geometric interpretation to this property.
2.2. REMARK. Let
\[ A_d(\delta, \sigma_1, \sigma_2) = \{ d \tau(\theta) \mid \delta + \sigma_1 \leq \theta \leq \delta + \sigma_2 \}, \]
be the arc of circle centered in the origin, radius \( d \), angle of the reference axis \( \delta \), aperture \( \sigma_2 - \sigma_1 \) and let,
\[ A_\epsilon( \delta, \sigma_1, \sigma_2) = \{ z \in \mathbb{R}^2 \mid \exists z' \in A_d(\delta, \sigma_1, \sigma_2) : \| z' - z \| < \epsilon \}, \]
be the \( \epsilon \)-neighborhood of \( A_d(\delta, \sigma_1, \sigma_2) \). Since, \( \forall t \geq 0, \)
\[ P_1(t) = P_0(t) - d_1 \tau(\theta_1(t) + \phi_1) - E_1(t) = P_0(t) + d_1 \tau(\theta_1(t) + \phi_1 + \pi) - E_1(t) \]
\[ = P_0(t) + d_1 \tau((\theta_0(t) + \phi_1 + \pi) - (\theta_1(t) - \theta_0(t)) - E_1(t), \]
we immediately deduce from (2.11) and (2.14) that the following property holds true,
\[ \forall \epsilon > 0, \exists \bar{t} : \forall t \geq \bar{t}, \ P_1(t) \in P_0(t) + A_\epsilon(\theta_0(t) + \phi_1 + \pi, \]
\[ - \arcsin(K_0^+ d_1 \cos \phi_1), - \arcsin(K_0^- d_1 \cos \phi_1)). \]
In other words, even if \( P_1 \) is not in general fixed in the leader’s reference frame \( \{ \tau(\theta_0(t)), \nu(\theta_0(t)) \} \)
during its motion, given any \( \epsilon > 0, P_1(t) \) belongs eventually to the \( \epsilon \)-neighborhood of the arc of circle,
\[ A_d(\theta_0(t) + \phi_1 + \pi, - \arcsin(K_0^+ d_1 \cos \phi_1), - \arcsin(K_0^- d_1 \cos \phi_1)), \]
which is fixed with respect to the reference frame of robot $R_0$ (see Fig. 2.6, where $\sigma_1 = -\arcsin(K_0^- d_1 \cos \phi_1)$, $\sigma_2 = -\arcsin(K_0^- d_1 \cos \phi_1)$ and $\delta = \theta_0(t) + \phi_1 + \pi$ in the case of $K_0^- < 0 < K_0^+$).

2.3. REMARK. In (Consolini et al. 2008, Theorem 1), it has been proved that conditions (2.8) and (2.10) with the weak inequalities are necessary and sufficient for the existence of controls that satisfy the constraints and maintain the vehicles exactly in $(d_1, \phi_1, 0)$-formation, that is $E_1(t) = 0, \forall t \geq 0$.

2.4 Stabilization of a multirobot formation

In this section we extend the results presented in Sect. 2.3 to multirobot formations. The following definitions are introduced to provide concise statements in Theorem 2.1.

2.6. DEFINITION. Let $(D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n$ be an admissible triple. For every $i = 0, \ldots, n$, let,

$$\mathcal{L}_i = \{ j \mid i < j \leq n \text{ and } l_j = i \},$$

be the set of the indexes $j$ such that $R_i$ is the relative leader of $R_j$. Recall that $\forall i = 1, \ldots, n$, $l_i$ denotes the index of the robot which is the leader of $R_i$.

Remark that $R_i$ is a relative leader if and only if $\mathcal{L}_i \neq \emptyset$ and it is always $\mathcal{L}_0 \neq \emptyset$ and $\mathcal{L}_n = \emptyset$.

2.7. DEFINITION. Let $K_i^-, K_i^+, i = 1, \ldots, n$ be $2n$ real constants and set $\forall i = n, n-1, \ldots, 1$,

$$\hat{K}_i^- = \max\{ K_i^-, \chi_{d_i, \phi_j}^{-1}(\tilde{K}_j^-) \mid j \in \mathcal{L}_i \},$$

$$\hat{K}_i^+ = \min\{ K_i^+, \chi_{d_i, \phi_j}^{-1}(\tilde{K}_j^+) \mid j \in \mathcal{L}_i \},$$

and

$$\hat{K}_0^- = \max\{ \chi_{d_i, \phi_j}^{-1}(\tilde{K}_j^-) \mid j \in \mathcal{L}_0 \},$$

$$\hat{K}_0^+ = \min\{ \chi_{d_i, \phi_j}^{-1}(\tilde{K}_j^+) \mid j \in \mathcal{L}_0 \}.$$ 

Remark that $\hat{K}_i^\pm = K_i^\pm$ if $\mathcal{L}_i = \emptyset$, therefore $\hat{K}_n^- = K_n^+$. Moreover, it is $\hat{K}_i^- \leq \hat{K}_i^+$ if $K_i^- \leq K_i^+$, by the monotonicity of $\chi^{-1}$.

2.8. DEFINITION. Let $(D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n$ be an admissible triple. Suppose that, $\forall i = 1, \ldots, n-1$,

$$\begin{cases} 
\text{if } \phi_i < 0, & -\frac{1}{d_i \cos \phi_i} \\
\text{if } \phi_i \geq 0, & -\frac{1}{d_i} 
\end{cases} < K_i^- \leq K_i^+ < \begin{cases} 
\frac{1}{d_i \cos \phi_i}, & \text{if } \phi_i > 0 \\
\frac{1}{d_i}, & \text{if } \phi_i \leq 0
\end{cases} \quad (2.21)$$
is satisfied. Then \( \forall \, i = 0, \ldots, n \) it is:

\[
-\pi/2 < \arcsin(K_i^- d_i \cos \phi_i) - \phi_i < \arcsin(K_i^+ d_i \cos \phi_i) - \phi_i < \pi/2.
\]

Therefore the following constants are well defined \( \forall \, i = n, n-1, \ldots, 1 \),

\[
\tilde{V}_i = \min\{V_i, \tilde{V}_j \cos \phi_j (\cos(0 \wedge (\arcsin(K_j^+ d_j \cos \phi_j) - \phi_j)) \wedge (\phi_j - \arcsin(K_j^- d_j \cos \phi_j)))^{-1} | j \in \mathcal{L}_i\},
\]

and

\[
\tilde{V}_0 = \min\{\tilde{V}_j \cos \phi_j (\cos(0 \wedge (\arcsin(K_0^+ d_j \cos \phi_j) - \phi_j)) \wedge (\phi_j - \arcsin(K_0^- d_j \cos \phi_j)))^{-1} | j \in \mathcal{L}_0\}.
\]

Remark that \( \tilde{V}_i = V_i \) if \( \mathcal{L}_i = \emptyset \), therefore \( \tilde{V}_n = V_n \). In the following, we will refer to the expressions in Definitions 2.7 and 2.8 as constraints propagation formulae.

We are now ready to deal with Problem 2.1 and state the main theorem of this chapter. Given an admissible triple \((D, \Phi, \mathcal{N})\) and \( n + 1 \) robots \( \mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_m \) with preassigned velocity and curvature constraints, the following theorem gives a method to determine corresponding bounds on the velocity and curvature of the main leader \( \mathbf{R}_0 \), such that the robots asymptotically achieve the desired formation while respecting their constraints at all times.

**2.1. Theorem (Main Theorem).** Let \((D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n\) be an admissible triple, \( \mathbf{R}_0, \ldots, \mathbf{R}_n \) \( n + 1 \) robots and \( V_i > 0, K_i^- \leq K_i^+, i = 0, \ldots, n, 3(n+1) \) real constants such that \( \forall \, i = 1, \ldots, n \), condition (2.21) is satisfied. Suppose that there exists a constant \( W_0 \) such that,

\[
0 < W_0 \leq v_0(t) \leq V_0, \quad K_0^- \leq \kappa_0(t) \leq K_0^+.
\] (2.22)

Let \( \tilde{K}_0^-, \tilde{K}_0^+, \tilde{V}_0 \) be as in Definitions 2.7 and 2.8. If the following conditions are satisfied,

\[
\tilde{K}_0^- < K_0^- \leq K_0^+ < \tilde{K}_0^+, \quad V_0 < \tilde{V}_0,
\] (2.23)

then there exists \( 2n \) suitable controls \( v_i, \omega_i, i = 1, \ldots, n \) such that the robots \( \mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_n \) are asymptotically in \((D, \Phi, \mathcal{N})\)-formation with leader \( \mathbf{R}_0 \) and the following conditions are verified,

\[
0 < W_i \leq v_i(t) \leq V_i, \quad K_i^- \leq \kappa_i(t) \leq K_i^+, \quad \forall \, t \geq 0,
\] (2.24)

where \( W_i \) is given in (2.28). Moreover \( \forall \, \epsilon > 0, \exists \tilde{t} : \forall \, t \geq \tilde{t}, \)

\[
P_i(t) \in P_i(t) + \mathcal{A}_d^*(\theta_i(t) + \phi_i + \pi, -\arcsin(K_i^+ d_i \cos \phi_i), -\arcsin(K_i^- d_i \cos \phi_i)), \quad \forall \, i = 1, \ldots, n.
\] (2.25)
Proof: The proof relies on the iterative application of Proposition 2.1. Define the following constants,

\[ \hat{K}_0^- = K_0^-, \quad \hat{K}_0^+ = K_0^+, \quad \hat{K}_i^- = \chi_{d_i,\phi_i}(\hat{K}_i^-), \quad \hat{K}_i^+ = \chi_{d_i,\phi_i}(\hat{K}_i^+). \]

We want to verify that \( \forall i = 1, \ldots, n, \)

\[ \hat{K}_i^- < \hat{K}_i^-, \quad \hat{K}_i^+ < \hat{K}_i^+. \tag{2.26} \]

Remark that the interval \([\hat{K}_i^-, \hat{K}_i^+]\) represents the curvature range necessary to the \(i\)-th robot to follow its relative leader. On the other hand, the interval \([\hat{K}_i^-, \hat{K}_i^+]\) represents the curvature range that is compatible with the curvature constraints of the robots contained in the subtree having the \(i\)-th robot as root. We will prove only the second inequality in (2.26) since the first one follows analogously. Set \( i = 1 \), by hypothesis (2.23) and monotonicity of \( \chi_{d_1,\phi_1} \),

\[ \hat{K}_1^+ = \chi_{d_1,\phi_1}(\hat{K}_0^+) = \chi_{d_1,\phi_1}(K_0^+) < \chi_{d_1,\phi_1}(\hat{K}_0^-) = \hat{K}_1^- \]

By induction, let suppose that the second in (2.26) holds for \( 1 \leq i \leq m - 1 \) (with \( 1 < m \leq n \)). Since the property holds for \( i = l_m \), being \( 1 \leq l_m \leq m - 1 \), we have as before,

\[ \hat{K}_m^+ = \chi_{d_m,\phi_m}(\hat{K}_m^+) < \chi_{d_m,\phi_m}(\hat{K}_m^-) \leq \chi_{d_m,\phi_m}(\hat{K}_m^-) = \hat{K}_m^- \]

Therefore, the second in (2.26) holds for \( i = m \). For any \( \rho > 0 \), define,

\[ \hat{K}_{0,\rho}^- = K_0^-, \quad \hat{K}_{0,\rho}^+ = K_0^+, \]

and \( \forall i = 1, \ldots, n, \)

\[ \hat{K}_{i,\rho}^- = \chi_{d_i,\phi_i}(\hat{K}_{i,\rho}^- - \rho), \quad \hat{K}_{i,\rho}^+ = \chi_{d_i,\phi_i}(\hat{K}_{i,\rho}^+ + \rho). \]

By the continuity of \( \chi_{d_i,\phi_i} \) we get that,

\[ \lim_{\rho \to 0} \hat{K}_{i,\rho}^\pm = \hat{K}_i^\pm, \quad \forall i = 1, \ldots, n. \]

Therefore, by (2.26) there exists a \( \rho > 0 \), such that, \( \forall i = 1, \ldots, n, \)

\[ \hat{K}_i^- < \hat{K}_{i,\rho}^- \leq \hat{K}_{i,\rho}^+ < \hat{K}_i^+. \tag{2.27} \]

Let \( \epsilon > 0 \) be sufficiently small, such that,

\[ \hat{K}_i^- < \hat{K}_{i,\rho}^- - \epsilon < \hat{K}_{i,\rho}^+ \leq \hat{K}_{i,\rho}^+ < \hat{K}_{i,\rho}^+ + \epsilon < \hat{K}_i^+, \]
2.4. Stabilization of a multirobot formation

and set, $\forall i = 1, \ldots, n$,

$$W_i = \frac{W_i}{2 \cos \phi_i} \cos \left( (\arcsin((\hat{K}_{i,\rho}^+ + \epsilon) d_i \cos \phi_i) - \phi_i) \vee (\phi_i - \arcsin((\hat{K}_{i,\rho}^- - \epsilon) d_i \cos \phi_i)) \right).$$

(2.28)

Now we prove (2.24) and (2.25) by induction. First of all, take $i = 1$; in this case (2.21) is exactly (2.8). If we apply Proposition 2.1 with $K^+_i = \hat{K}_{i,\rho}^+$, remarking that $\hat{K}^+_0$ in Proposition 2.1 is just $\chi^{-1}_{d_i,\phi_i}(K^+_i)$, we get that,

$$\hat{K}_0^+ = \chi^{-1}_{d_1,\phi_1}(\hat{K}_{1,\rho}^+) = \chi^{-1}_{d_1,\phi_1}(\chi_{d_1,\phi_1}(\hat{K}_{0,\rho}^\pm) = K^+_0 \pm \rho,$$

which implies condition (2.10). Therefore by Proposition 2.1, we have that,

$$\lim_{t \to +\infty} E_1(t) = 0,$$

and

$$0 < W_1 \leq v_1(t) \leq V_1,$$

$$K^-_1 \leq \hat{K}_1^- < \hat{K}_{1,\rho}^- \leq \kappa_1(t) \leq \hat{K}_{1,\rho}^+ < \hat{K}_1^+ \leq K^+_1.$$

This means that (2.24) is verified for $i = 1$. Let suppose by induction that (2.24) holds for $1 \leq i \leq m - 1$ (with $1 < m \leq n$) and take, in Proposition 2.1, $K^+_i = \hat{K}_{i,\rho}^+$ and $K^-_i = \hat{K}_{i,\rho}^-$. Therefore $\hat{K}_0^+$ in Proposition 2.1 is given by,

$$\hat{K}_0^+ = \chi^{-1}_{d_m,\phi_m}(\hat{K}_{m,\rho}^+) = \chi^{-1}_{d_m,\phi_m}(\chi_{d_m,\phi_m}(\hat{K}_{0,\rho}^\pm) = \hat{K}_{m,\rho}^+ \pm \rho.$$

Moreover, by (2.21), (2.27) and the inductive hypotheses, it is,

$$\begin{cases} \text{if } \phi_m < 0, \ & -\frac{1}{d_m \cos \phi_m} \\ \text{if } \phi_m \geq 0, \ & -\frac{1}{d_m} \end{cases} \leq \begin{cases} K^-_{l_m} \leq \hat{K}_{l_m}^- < \hat{K}_{l_m,\rho}^- \leq \kappa_{l_m}(t) \leq \hat{K}_{l_m,\rho}^+ < \hat{K}_{l_m}^+ \leq K^+_{l_m} \\ \frac{1}{d_m \cos \phi_m}, \ & \text{if } \phi_m > 0 \\ \frac{1}{d_m}, \ & \text{if } \phi_m \leq 0. \end{cases}$$

Therefore all the hypotheses of Proposition 2.1 are verified if we take in Proposition 2.1, $K^+_0 = \hat{K}_{m,\rho}^+$, $K^+_1 = \hat{K}_{m,\rho}^+$ and $\hat{K}^-_0 = \hat{K}_{m,\rho}^- \pm \rho$. Then applying it again, we obtain that (2.24) holds for $i = m$ and $\lim_{t \to +\infty} E_m(t) = 0$. Finally, (2.25) is a straight consequence of Remark 2.2 applied to each leader-follower unit $(R_{l_i}, R_i)$.

2.4. Remark. As shown in the proof above, under the hypotheses of Theorem 2.1, if we set

$$\Gamma^+ = \{ \gamma \in S^1 | (K^-_{l_i} - \epsilon) d_i \cos \phi_i \leq \sin \gamma \leq (K^+_{l_i} + \epsilon) d_i \cos \phi_i \},$$
and \( \beta_i(t) = \theta_i(t) - \theta_i(t) \), the controls,

\[
v_i(t) = \begin{cases} 
V_i, & \text{if } \beta_i(t) \notin \Gamma_i \\
v_i(t) \cos(\beta_i(t) - \phi_i) + \eta_i(t) \langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle / \cos \phi_i, & \text{if } \beta_i(t) \in \Gamma_i
\end{cases}
\]

\[
\omega_i(t) = \begin{cases} 
V_i K_i^+, & \text{if } \beta_i(t) \notin \Gamma_i \text{ and } K_i^+ \geq 0 \\
V_i K_i^-, & \text{if } \beta_i(t) \notin \Gamma_i \text{ and } K_i^+ < 0 \\
v_i(t) \sin \beta_i(t) + \eta_i(t) \langle E_i(t), \nu(\theta_i(t)) \rangle / d_i \cos \phi_i, & \text{if } \beta_i(t) \in \Gamma_i
\end{cases}
\]

This system of equations assures that \( R_0, R_1, \ldots, R_n \) are asymptotically in \((D, \Phi, \mathcal{N})\)-formation with leader \( R_0 \). \( \eta_i(t) \) is a bounded continuous function given by,

\[
\eta_i(t) = \frac{(v_i(t) - W_i/2) \cos(\beta_i(t) - \phi_i)}{|\langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle|} \wedge \frac{((K_i^+ + \epsilon/2 - \kappa_i(t)) \wedge (K_i^- - \epsilon/2)) d_i \cos \phi_i}{|E_i(t), \nu(\theta_i(t))|} \\
\wedge \frac{V_i \cos \phi_i - v_i(t) \cos(\beta_i(t) - \phi_i)}{|\langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle|} \wedge \frac{v_i(t) (K_i^+ d_i \cos(\beta_i(t) - \phi_i) - \sin \beta_i(t))}{|\langle E_i(t), \nu(\theta_i(t)) \rangle| + |K_i^+| |\langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle|} \\
\wedge \frac{v_i(t) \sin \beta_i(t) - K_i^- d_i \cos(\beta_i(t) - \phi_i)}{|\langle E_i(t), \nu(\theta_i(t)) \rangle| + |K_i^-| |\langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle|} \wedge M_i,
\]

where \( M_i \) are positive gain constants.

### 2.5 Simulation experiments

Simulation experiments have been conducted to illustrate the theory and show the effectiveness of the formation controller presented in Sect. 2.4.

We considered the \((D, \Phi, \mathcal{N})\)-formation reported in Fig. 2.3(a), with \( d_1 = 2 \text{ m}, d_2 = d_3 = 1 \text{ m} \) and \( \phi_1 = 4\pi/9 \text{ rad}, \phi_2 = \phi_3 = \pi/6 \text{ rad} \). The initial conditions of the robots are \( R_0 = (5, 2.5, \pi/3)^T, R_1 = (7, -1, \pi/4)^T, R_2 = (4.5, 0, \pi/10)^T \) and \( R_3 = (5.8, -0.5, \pi/6)^T \). The main leader \( R_0 \) moves along a circular path, with velocities \( v_0(t) = 1.2 \text{ m/s}, \omega_0(t) = 0.25 \text{ rad/s} \). We chose \( W_0 = 1 \text{ m/s}, V_0 = 1.5 \text{ m/s}, V_1 = V_2 = V_3 = 3 \text{ m/s} \) and \( K_0^+=-0.25 \text{ rad/m}, K_0^- = 0.5 \text{ rad/m}, K_1^- = K_2^- = K_3^- = -0.5 \text{ rad/m}, K_1^- = K_2^- = K_3^- = 1 \text{ rad/m} \). With these values condition (2.21) is satisfied for \( i = 1, 2, 3 \) and inequalities (2.22) hold true. Note that in this case \( \mathcal{L}_0 = \{1, 2\}, \mathcal{L}_1 = \mathcal{L}_3 = \emptyset \) and \( \mathcal{L}_2 = \{3\} \). Using the constraints propagation formulae in Definitions 2.7 and 2.8, we get the bounds \( \tilde{K}_0^-, \tilde{K}_0^+, \tilde{V}_0 \). It is a simple task to verify that \( \tilde{K}_0^-, \tilde{K}_0^+ \) and \( \tilde{V}_0 \) satisfy condition (2.23). The gains \( M_1, M_2 \) and \( M_3 \) in (2.29) were set to 1. Fig. 2.8(a) shows the trajectory of the robots and the arcs of circle \( \mathcal{A}_{d_i}^l(\theta_i(t) + \phi_i + \pi, -\arcsin(K_i^+ d_i \cos \phi_i), -\arcsin(K_i^- d_i \cos \phi_i)) \),
\(i = 1, 2, 3\) defined in Remark 2.2 \((\varepsilon = 10^{-3})\). In order to have a temporal reference in the figure, the vehicles are drawn every two seconds. Note that after about 9 seconds the robots achieve the desired formation. Fig. 2.8(b) shows that the norm of the errors \(E_1\), \(E_2\) and \(E_3\) asymptotically converges to zero. Finally, Fig. 2.8(c) reports the time history of the angles \(\beta_i\) (solid) and the bounds \(\arcsin(K_i \pm d_i \cos \phi_i)\) (dash). Except for the early instants, the angles \(\beta_i\) keep always inside the respective bounds, thus confirming property (2.25).

In order to test the robustness of the proposed formation control scheme, in a second simulation experiment (see Figs. 2.9(a)-2.9(c)) the position and angular measurements of the four robots have been corrupted with white Gaussian noise with zero mean and variance \(0.5 \times 10^{-3}\). Fig. 2.9(a) shows the trajectory of the robots and the arcs of circle defined in Remark 2.2. Despite the noisy data, the robots reach the desired formation, \(v_i, \kappa_i\) still satisfy condition (2.24), the norm of the errors \(E_i\) converges asymptotically to zero (Fig. 2.9(b)) and the angles \(\beta_i\) keep inside the respective bounds (Fig. 2.9(c)).

### 2.6 Extensions and further results

Some variations to the basic setup studied in the previous sections have been proposed in (Morbidi and Prattichizzo 2007, Morbidi et al. 2007).

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**Figure 2.7:** Extension (more involved kinematic models): the articulated vehicle (leader) and the car-like robot (follower) achieve a time-varying desired formation.
In (Morbidi and Prattichizzo 2007), formations of nonholonomic vehicles with more involved kinematics have been considered. In fact the follower is herein supposed to be a car-like robot and the leader an articulated vehicle, a tractor pulling a trailer. The leader moves along an assigned trajectory while the follower is to maintain a (time-varying)
2.6. Extensions and further results

Figure 2.9: Noisy data: (a) Trajectory of the robots and arcs of circle $A'_{d_i}(\theta_i(t) + \phi_i + \pi, -\arcsin(K^+_{d_i} d_i \cos \phi_i), -\arcsin(K^-_{d_i} d_i \cos \phi_i))$, $i = 1, 2, 3$. (b) Norm of the errors $E_1$, $E_2$ and $E_3$; (c) Angles $\beta_i$ (solid) and bounds $\arcsin(K^\pm_{d_i} d_i \cos \phi_i)$ (dash), $i = 1, 2, 3$.

desired distance and orientation to the trailer (see Fig. 2.7). Our personal motivation for such an unusual formation stemmed from a possible real-world application: the control of truck-trailer, car platoons in automated highway systems (AHSs) (Swaroop and Hedrick 1999). A sliding mode formation control scheme (Utkin 1977) is proposed.
to asymptotically stabilize the vehicles to a time-varying desired formation. The attitude angles of the follower and the tractor are estimated via global exponential observers based on the Immersion and Invariance technique (Karagiannis et al. 2008) (for more details on this technique, see Chapter 3, Sect. 3.7).

In (Morbidi et al. 2007), the leader-follower formation control of nonholonomic mobile robots is presented as a disturbance decoupling problem (Basile and Marro 1992). By means of pure geometrical arguments (Isidori et al. 1981, Nijmeijer and Van der Schaft 1983), it is shown that the formation control problem is solvable if the velocity vector of the leader can be measured and a scalable (with the number of followers) solution to the problem is proposed. Actually, the key idea of the work is to interpret the control input of the leader as a disturbance affecting the state of the leader-follower system.

2.7 Conclusions and future work

In this chapter we have presented a new geometric approach to the asymptotic stabilization of leader-follower formations of unicycle robots with input constraints. The effect of these input bounds on the admissible trajectories of the leader that guides the formation is studied and recursive formulae for the maximum velocity and curvature of the main leader, such that all the other robots can follow their relative leaders while respecting their input constraints, are provided. A new general class of multirobot formations, called hierarchical formations, is introduced. Some extensions and further results are briefly discussed at the end of the chapter. Simulation experiments illustrate the theory and show the effectiveness of the proposed designs.

The study of a decentralized version of our formation control strategy and the inclusion of communication/visibility constraints among the robots, are subjects of on-going research.
Chapter 3

Vision-based localization of robot formations

A subtle thought that is in error may yet give rise to fruitful inquiry that can establish truths of great value.

I. Asimov

Abstract

This chapter deals with vision-based localization for leader-follower formation control. Each unicycle robot is equipped with a panoramic camera which only provides the view-angle to the other robots. As an original contribution, the vision-based localization problem is analytically studied here using a new observability condition valid for general nonlinear systems and based on the Extended Output Jacobian. This condition is attractive since it allows one to identify the set of leader trajectories that preserve system’s observability. The state of the leader-follower system is estimated via the extended Kalman filter (EKF) and an input-state feedback controller is designed to stabilize the formation. Some extensions to the setup are discussed at the end of the chapter: we study how the proposed approach can be adapted to deal with the critical case of distant robots and analyze the closed-loop stability of the system when the EKF is replaced by a novel reduced-order observer based on the Immersion and Invariance (I&I) technique. Simulations and real-data experiments have been performed to test the effectiveness and robustness of the proposed designs.

The material of this section is based on (Mariottini et al. 2007, Morbidi, Mariottini and Prattichizzo 2008, Mariottini et al. 2009, Conditionally accepted, Morbidi et al. 2009, Conditionally accepted).

3.1 Introduction

Cameras have recently played a leading role in robotics due to their low cost and to the rich information they provide when compared to other traditional sensors (e.g. laser range finders, sonars). Solving the formation control problem for a group of multiple robots using exclusively on-board vision sensors is challenging because cameras only provide the view-angles to the other robots, but not the distance. In this respect,
the formation can be controlled only if a *localization problem* has been solved, i.e. only the data provided by the on-board cameras are used to estimate the relative distance of the robots w.r.t. a common reference frame (e.g. on the leader, in a leader-follower approach).

For the localization problem to be solvable, the system should be *observable*. From control theory, a system is observable when it is possible to reconstruct the initial state by knowing, in a given time interval, the control inputs and the sensor measurements. In particular, when dealing with vision sensors, the localization is intrinsically nonlinear (Bicchi et al. 1998), meaning that linearized approximations can be non-observable while tools from nonlinear systems theory prove the possibility to reconstruct the state. Such a problem is usually referred to as the *observability of perspective dynamical systems* and it recently stimulated a great deal of research both in the robotics and control communities (Hespanha 2002, Dixon et al. 2003, Chen and Kano 2002, Abdursul et al. 2004, Hu et al. 2008).

In (Conticelli et al. 2000), the state estimation problem for a single robot with on-board camera is approached using a Luenberger-like nonlinear observer based on the projection of stationary landmarks in the environment. In (Vidal et al. 2004), the localization problem for a team of nonholonomic mobile robots with calibrated vision sensors has been solved using motion segmentation techniques based on optical-flow. In (Das et al. 2002), an interesting centralized framework for vision-based leader-follower formation control has been introduced. However, since both the camera calibration parameters and the height of the camera to the floor are supposed a priori known, then the distance between the robots can be computed directly by the panoramic cameras. These strong assumptions restrict the practical applicability of the control strategy in (Das et al. 2002) to near robots.

In this chapter, we consider leader-follower formations of unicycle robots equipped only with omnidirectional cameras.

As an original contribution, the localization problem is analytically studied using a *new* observability condition valid for general nonlinear systems and based on the Extended Output Jacobian. As an improvement over the existing literature, we do not assume to know any camera calibration parameter (mirror shape or focal length), nor the pose of any stationary landmark: only the view-angle to the other robots is provided by each camera, but not the distance, that is estimated by a nonlinear observer (the extended Kalman filter (EKF)). An input-state feedback control law is designed to stabilize the formation.

As a second contribution, thanks to our observability condition, we identify those trajectories of the leader that preserve system observability. An insightful geometrical interpretation of vision-based formation localizability is presented and the validity range of our observability condition is discussed.
3.2 Leader-follower setup

At the end of the chapter further results are presented. We first show how the proposed input-state feedback control strategy can be adapted to address the critical case of distant robots. Second, we design a reduced-order observer alternative to the EKF, for leader-follower range estimation. The new estimator relies on the Immersion and Invariance (I&I) technique (Karagiannis et al. 2008), it provides a globally asymptotically convergent estimate of the range, it can be easily tuned to achieve the desired convergence rate by acting on a single gain parameter and it is extremely simple to implement as well. The stability of the closed-loop system arising from the combination of the I&I range estimator and the formation controller presented in the first part of the chapter, is proved via Lyapunov arguments.

Simulations and real-data experiments have been conducted to validate the theory and show the robustness of the proposed designs.

The rest of the chapter is organized as follows. In Sect. 3.2, the leader-follower kinematic model and the assumptions on sensing and communication are presented. In Sect. 3.3, we introduce the new observability condition based on the Extended Output Jacobian. In Sect. 3.4, the input-state feedback control law is designed. Simulations as well as experimental results are presented and discussed in Sects. 3.5.1 and 3.5.2. In Sects. 3.6, 3.7, additional results on formation control and range estimation are introduced. In Sect. 3.8 the main contributions of the chapter are summarized and future research directions are highlighted. The Appendix reviews some basic facts on the consistency of a state estimator.

3.2 Leader-follower setup

3.2.1 Kinematic model

Let us consider the leader-follower setup of Fig. 3.1. The kinematics of each robot can be abstracted as a unicycle model,

\[
\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = \omega,
\]

where \((x, y)\) represents the position of each robot and \(\theta\) its orientation with respect to the world frame W. The leader L has a configuration vector \((x_L, y_L, \theta_L)^T\) while the follower F has a vector \((x_F, y_F, \theta_F)^T\). The control inputs of the leader and the follower are the linear and angular velocities \((v_L, \omega_L)^T\) and \((v_F, \omega_F)^T\), respectively.

The whole leader-follower system can be modelled using polar coordinates, where \(\rho\) is the distance from the center of the leader to a marker \(P\) placed at a known distance \(d\) on the follower (see Fig. 3.1). The variable \(\psi\) is the view-angle from the y-axis of the leader to the marker \(P\), while \(\beta\) is the relative orientation of the robots, i.e., \(\beta \triangleq \theta_L - \theta_F\).
In the spirit of (Das et al. 2002, Mariottini et al. 2005), we introduce here the following kinematic model:

3.1. PROPOSITION (LEADER – FOLLOWER KINEMATIC MODEL). With reference to Fig. 3.1, the leader – follower kinematic model can be written as follows:

\[ \dot{s} = G(s)u, \]  

where \( s \triangleq (\rho, \psi, \beta)^T \), \( u \triangleq (v_F, \omega_F, v_L, \omega_L)^T \) and,

\[
G(s) = \begin{bmatrix}
\cos \gamma & d \sin \gamma & -\cos \psi & 0 \\
-\sin \gamma & d \cos \gamma & \sin \psi & 0 \\
\frac{\rho}{\rho} & \frac{\rho}{\rho} & 0 & 1 \\
0 & -1 & 0 & 1
\end{bmatrix},
\]

where \( \gamma \triangleq \beta + \psi \).

The kinematic model in the case of \( q \) followers can be obtained by simply extending (3.2). In this case the input vector is \( u \triangleq (v_{F_1}, \omega_{F_1}, \ldots, v_{F_q}, \omega_{F_q}, v_L, \omega_L)^T \) and the state vector is \( s \triangleq (s_1^T, s_2^T, \ldots, s_q^T)^T \in \mathbb{R}^{3q} \) (see (Mariottini et al. 2005) for more details).

3.2.2 Sensing

Each robot is equipped only with an omnidirectional camera (Benosman and Kang 2001). This sensor is particularly suited for mobile robot navigation, due to its field of view that

Figure 3.1: Leader-follower setup in polar coordinates representation.
3.2. Leader-follower setup

![Image of leader-follower setup]

Figure 3.2: View-angles computation (on the leader): HSV color blob detection is used to determine the two angles $\xi$ and $\psi$ from the leader to the follower’s center and marker, respectively.

is wider than that of a standard pinhole camera (see Fig. 3.2). According to the setup in Fig. 3.1, L can measure the view-angles $\xi$ and $\psi$ given by the observation of the follower’s centroid and the colored marker $P$, respectively. Analogously, the camera on F can measure the view-angle $\eta$ to the leader.

The measurement of view-angles is obtained on each robot by means of an automatic real-time color detection and tracking algorithm (Forsyth and Ponce 2002). Since in our setup each robot and the landmark $P$ have been characterized by a specific color (see also Fig. 3.6), the leader-follower sensing association is fully automatic. More implementation details will be discussed in Sect. 3.5.2.

3.2.3 Communication

The state estimation process and the control computation are centralized on the leader, which transmits to the follower the control velocities $(v_F, \omega_F)^T$ needed to drive the formation (see Fig. 3.3). Due to the above assumption, the inter-robot communication is made fast because the follower will only need to transmit its view-angle $\eta$ to the leader. We assume no communication delays in the view-angle transmission.

From Fig. 3.1 is evident that $\beta$ can be computed as follows:

$$\beta = -\xi + \eta + \pi. \quad (3.3)$$

To simplify the discussion, we will henceforth refer only to $\beta$, implicitly assuming the transmission of $\eta$. To summarize, the leader measures a two dimensional output vector,

$$y \triangleq (y_1, y_2)^T = (\psi, \beta)^T. \quad (3.4)$$
3. Vision-based localization of robot formations

As a concluding remark we emphasize here two original contributions which differentiate the present work from (Das et al. 2002). First of all, we do not assume a full knowledge of camera calibration parameters. In fact, only the image center \( C \) is needed to compute the view-angles and, in many practical cases, \( C \) simply coincides with the central black hole in panoramic images (see Fig. 3.2). Second, and most important, we assume that the leader-follower relative distance \( \rho \) in (3.2) is unknown. The problem of range estimation will be studied in detail in the next section.

3.3 Vision-based observability of robot formations

3.3.1 Observability of nonlinear systems

This section reviews some basic facts about the observability of nonlinear systems (Inaba et al. 2000, Isidori 1995) and presents in Proposition 3.3 a new rank condition, that will be used to study the observability of vision-based leader-follower formations.

Consider a generic nonlinear system \( \Sigma \) of the form

\[
\Sigma : \begin{cases}
\dot{s}(t) = f(s(t), u(t)), & s(0) \equiv s^0 \\
y(t) = h(s(t)) = (h_1, h_2, \ldots, h_m)^T
\end{cases}
\]  

(3.5)

where \( s(t) = (s_1(t), s_2(t), \ldots, s_n(t))^T \in \mathcal{S} \) is the state, \( y(t) \in \mathcal{Y} \) the measurements vector and \( u(t) \in \mathcal{U} \) the input vector. \( \mathcal{S} \), \( \mathcal{Y} \) and \( \mathcal{U} \) are differential manifolds of dimension \( n \), \( m \) and \( p \), respectively. The problem of observability for \( \Sigma \) can be roughly viewed as the injectivity of the input-output map \( \mathcal{R}_\Sigma : \mathcal{S} \times \mathcal{U} \rightarrow \mathcal{Y} \) with respect to the initial conditions.
3.3. Vision-based observability of robot formations

Let \( y(t, s_0, u) \triangleq h(s(t, s_0, u)) \) denote the output of \( \Sigma \) at time \( t \), for input \( u \) and initial state \( s(0) = s_0 \). Two states \( s_1, s_2 \in S \) are said to be indistinguishable (denoted \( s_1 I s_2 \)) for \( \Sigma \) if for every admissible input function \( u \), the output function \( y(t, s_1, u), t \geq 0 \), of the system for initial state \( s_1 \), and the output function \( y(t, s_2, u), t \geq 0 \), of the system for initial state \( s_2 \), are identical on their common domain of definition (Nijmeijer and Van der Schaft 1990, p. 93). (If in addition the trajectories of the system emanating from \( s_1 \) and \( s_2 \) both lie in a subset \( M \) of \( S \), then \( s_1 \) and \( s_2 \) are said to be \( M \)-indistinguishable and denoted \( s_1 I_M s_2 \)). Note that the indistinguishability \( I \) is an equivalence relation on \( S \).

The notions of observability and indistinguishability are tightly connected, as shown in the next definition (Hermann and Krener 1977).

3.1. Definition (Observability). System \( \Sigma \) is said to be observable at \( s^0 \) if \( I(s^0) = \{ s^0 \} \).

For a generic nonlinear system as the one in (3.5), global or complete observability can not be usually expected and the more specific notion of local weak observability has been introduced in the literature (Hermann and Krener 1977, p. 733). This notion has also the advantage to lend itself to a simple algebraic test.

3.2. Definition (Local Weak Observability). \( \Sigma \) is said to be locally weakly observable at \( s^0 \) if there exists an open neighborhood \( M \) of \( s^0 \) such that for every open neighborhood \( V \) of \( s^0 \) contained in \( M \), \( I_V(s^0) = \{ s^0 \} \).

Intuitively, \( \Sigma \) is locally weakly observable if one can instantaneously distinguish each point from its neighbors. Before we proceed any further, let us introduce two differential operators that will be useful in the next derivations.

Given a scalar-valued function \( \lambda(s) : \mathbb{R}^n \to \mathbb{R} \), the gradient of \( \lambda \) is defined as follows:

\[
\frac{d\lambda(s)}{ds} \triangleq \begin{pmatrix}
\frac{\partial \lambda(s)}{\partial s_1} \\
\frac{\partial \lambda(s)}{\partial s_2} \\
\cdots \\
\frac{\partial \lambda(s)}{\partial s_n}
\end{pmatrix}
\]

The Lie derivative of a scalar-valued function \( h(s) \) along a vector field \( g : \mathbb{R}^n \to \mathbb{R}^n \) is a real-valued function, defined as \( L_g h(s) \triangleq dh(s) g \). The Lie derivative can be repeated recursively as,

\[
L_g^k h(s) \triangleq L_g \left( L_g^{k-1} h(s) \right), \quad \forall \ k \geq 1,
\]

being \( L_g^0 h(s) \triangleq h(s) \).

A sufficient condition for the local weak observability of \( \Sigma \) has been first introduced in (Hermann and Krener 1977, Theorem 3.1) and is here reported for the reader’s convenience.

3.2. Proposition (Observability Rank Condition). System \( \Sigma \) in (3.5) is locally weakly observable at a point \( s_0 \in S \), if there exist an open neighborhood \( M \) of \( s_0 \) and positive integers
3. Vision-based localization of robot formations

\(j_1, j_2, \ldots, j_m\) satisfying \(j_1 + j_2 + \ldots + j_m = n\) such that, for arbitrary \(s \in \mathcal{M}\), the observability matrix \(O\) defined as the matrix with rows,

\[
O \triangleq \{L_j^{-1} dh_i(s) \mid i = 1, \ldots, m; \ j = 1, \ldots, j_m\},
\]

is full rank.

An equivalent and more intuitive formulation of Proposition 3.2, that relies on the so-called Extended Output Jacobian (EOJ) (Conticelli et al. 2000), is presented in the next proposition.

3.3. Proposition (EOJ observability rank condition (Mariottini et al. 2005)). System \(\Sigma\) is locally weakly observable at a point \(s_0 \in S\), if there exist an open neighborhood \(\mathcal{M}\) of \(s_0\) such that, for arbitrary \(s \in \mathcal{M}\), the Extended Output Jacobian \(J \in \mathbb{R}^{mn \times n}\) defined as the matrix with rows,

\[
J \triangleq \{dh_i^{(j-1)}(s) \mid i = 1, \ldots, m; \ j = 1, \ldots, n\},
\]

is full rank. The superscript \(j\) refers to the order of time differentiation of the functions \(h_i(s)\).

Proof: The proof is constructive. Computing the Lie derivatives in (3.6), it turns out that for \(i = 1, \ldots, m\),

\[
(j = 1) \ L_0^0 dh_i(s) = \frac{\partial h_i(s)}{\partial s} = dh_i^{(0)}(s),
\]

\[
(j = 2) \ L_1^0 dh_i(s) = \frac{\partial}{\partial s} \left[L_0^0 dh_i(s)\right] f(s) =
\]

\[
= d \left[\frac{\partial h_i}{\partial s} f(s)\right] = d \left[\frac{\partial h_i}{\partial s} \frac{\partial s}{\partial t}\right] = dh_i^{(1)}(s),
\]

\[
(j = 3) \ L_2^0 dh_i(s) = d \left[\frac{\partial}{\partial s} \left[L_1^0 h_i\right] f(s)\right] =
\]

\[
= d \left[\frac{\partial h_i^{(1)}}{\partial s}\right] = d \left[\frac{\partial (\frac{\partial h_i}{\partial t})}{\partial s}\right] = dh_i^{(2)}(s),
\]

... ...

\[
(j = n) \ L_n^{n-1} dh_i(s) = dh_i^{(n-1)}(s),
\]

and by stacking the vectors (3.8)-(3.11) in a matrix, from Prop. 3.2 we obtain the thesis.

3.1. Remark. Proposition 3.3 states that the observability of \(\Sigma\) can be tested by checking the rank of the EOJ \(J\) made of the state partial derivatives of the output vector and of all its \(n - 1\) time derivatives. In particular, \(\Sigma\) is locally weakly observable also when at least one \(n \times n\) submatrix of \(J\) is full rank.
3.3.2 Observability of leader-follower formations

Proposition 3.3 is used here to determine an observability condition for the leader-follower system described in Sect. 3.2. From this proposition, the observability of (3.2) with output (3.4), is guaranteed when at least one $3 \times 3$ submatrix of the whole $6 \times 3$ EOJ is nonsingular. Let us consider, e.g. the submatrix $\bar{J}$:

\[
\bar{J} = \begin{bmatrix}
\frac{\partial y_1}{\partial \rho} & \frac{\partial y_1}{\partial \psi} & \frac{\partial y_1}{\partial \beta} \\
\frac{\partial y_2}{\partial \rho} & \frac{\partial y_2}{\partial \psi} & \frac{\partial y_2}{\partial \beta} \\
\frac{\partial y_3}{\partial \rho} & \frac{\partial y_3}{\partial \psi} & \frac{\partial y_3}{\partial \beta}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
\frac{\partial \dot{\psi}}{\partial \rho} & \frac{\partial \dot{\psi}}{\partial \psi} & \frac{\partial \dot{\psi}}{\partial \beta} \\
0 & 0 & 1
\end{bmatrix},
\]

whose determinant is,

\[
\det(\bar{J}) = -\frac{\partial \dot{\psi}}{\partial \rho} = \frac{1}{\rho} \left[ \dot{\psi} + \omega_L \right].
\]

Therefore, if $\dot{\psi} + \omega_L \neq 0$, the state $s$ is observable.

In the case of $q > 1$ followers, the observability condition is a simple extension of (3.13) (see (Mariottini et al. 2005)).

3.3.3 A geometrical interpretation of the EOJ singularity

In Fig. 3.4 we provide a basic example to give some geometrical insight into condition found in the previous section. A leader and two followers are considered at two different time instants, $t = 0$ and $t = 1$. All the robots move with the same translational velocity and zero angular velocity.

We first note that that $\psi_2(1) \neq \psi_2(0)$ (and thus $\dot{\psi}_2 \neq 0$) due to the different initial orientation between L and F$_2$. Then, from (3.13), it turns out that the state $s_2$ is observable. This is intuitively correct, since the visual information varies in time and it is then expected to improve the localizability. This leads also to the intuition that curvilinear trajectories have a favourable effect on observability, since a change of the output signal (3.4) occurs there.

From an inspection of Fig. 3.4, it is also evident that there is not any improvement in the localization between L and F$_1$ since their relative motion is zero from $t = 0$ to $t = 1$ (and thus $\dot{\psi}_1 = 0$). The state $s_1$ is then not observable. To confirm this intuition, we should verify to what extent the condition of Proposition 3.3 is also necessary. This point is discussed in full detail in the next section.

3.3.4 Necessity of the observability condition

Proposition 3.2 provides a sufficient condition for the local weak observability of $\Sigma$. However as pointed out in (Hermann and Krener 1977, Theorem 3.11), the converse
implication is “almost” true. This means that if $\Sigma$ is locally weakly observable, then the observability rank condition is satisfied generically. “Generically” means that the observability matrix $O$ in (3.6), and equivalently $J$ in (3.7), must be full rank everywhere except possibly within a subset of the domain of $s$ (Casti 1982). Algebraically, this is equivalent to test that $J$ is not full rank for all values of $s$, which will then imply that $\Sigma$ is not locally weakly observable. In the case of a single follower, the general expression for $J \in \mathbb{R}^{6 \times 3}$ is,

$$J^T = \begin{bmatrix} \frac{\partial y_1}{\partial p} & \frac{\partial y_1}{\partial \dot{p}} & \frac{\partial y_1}{\partial \ddot{p}} & \frac{\partial y_2}{\partial p} & \frac{\partial y_2}{\partial \dot{p}} & \frac{\partial y_2}{\partial \ddot{p}} \\ \frac{\partial y_1}{\partial v} & \frac{\partial y_1}{\partial \dot{v}} & \frac{\partial y_1}{\partial \ddot{v}} & \frac{\partial y_2}{\partial v} & \frac{\partial y_2}{\partial \dot{v}} & \frac{\partial y_2}{\partial \ddot{v}} \\ \frac{\partial y_1}{\partial \beta} & \frac{\partial y_1}{\partial \dot{\beta}} & \frac{\partial y_1}{\partial \ddot{\beta}} & \frac{\partial y_2}{\partial \beta} & \frac{\partial y_2}{\partial \dot{\beta}} & \frac{\partial y_2}{\partial \ddot{\beta}} \end{bmatrix}. \quad (3.14)$$

As also shown in Fig. 3.4, if $L$ and $F_1$ move along straight trajectories, then $y$ is constant, and hence $\dot{y} = \ddot{y} = 0$. From (3.14) this implies that $J$ has only two non-zero rows, i.e. its rank can never be 3. Therefore, the state $s_1$ of the system is not locally weakly observable exactly when the robots move along straight trajectories. For a system not to be observable, means that the output does not convey an information rich enough to allow the observer to provide a correct estimate of the state, thus negatively affecting the control action. Such an undesired effect on the control has been observed in both the simulation and experimental results, as discussed in Sects. 3.5.1-3.5.2.

### 3.3.5 Observer design

In order to control the formation, an estimate $\hat{s}$ of the true state $s$ is required. An extended Kalman filter has been designed to estimate the state $s$ given the input vector

![Diagram of robot formations and observability](image-url)
u and the output y. We assume additive noise on both the process and measurement equations,

\begin{align}
\dot{s} &= G(s)u + z \\
y &= Cs + v
\end{align}  \tag{3.15}  \tag{3.16}

where C is the output transition matrix and z and v are zero mean white Gaussian noises with covariance matrices Q and R, respectively. s(0), z and v are assumed to be uncorrelated. Equation (3.15) has been discretized using the Euler forward method with sampling time \( T_c \),

\[ s(k+1) = \Gamma(s(k), u(k)) + T_c z, \]

where \( \Gamma(s(k), u(k)) \triangleq T_c G(s)u + s(k) \) and \( k \in \mathbb{N} \).

### 3.4 Input-state feedback control

Consider the set of kinematic equations equivalent to (3.2):

\begin{align}
\dot{s}_r &= F(s)u_L + H(s)u_F \\
\dot{\beta} &= \omega_L - \omega_F
\end{align}  \tag{3.17}  \tag{3.18}

where \( s_r \triangleq (\rho, \psi)^T \) is the reduced state-space vector. \( H(s) \) and \( F(s) \) are the two upper left and right \( 2 \times 2 \) submatrices of \( G(s) \), respectively.

In the spirit of (Das et al. 2002), we propose here an input state feedback control law for the robot formation. Let us consider the following control input,

\[ u_F \triangleq (v_F, \omega_F)^T = H^{-1}(s)(p - F(s)u_L), \]  \tag{3.19}

where

\[ p = s_r^{des} - K(s_r - s_r^{des}), \]  \tag{3.20}

where \( K = \text{diag}\{k_1, k_2\} \), with \( k_1, k_2 > 0 \). The superscript “\( \text{des} \)” refers to the desired state. Equation (3.19) acts in (3.17) as a feedback linearizing control, so that the closed-loop dynamics becomes,

\[ \dot{s}_r = \dot{s}_r^{des} - K(s_r - s_r^{des}), \quad \dot{\beta} = \omega_L - \omega_F. \]  \tag{3.21}

The following proposition states that it suffices to control \( s_r \) towards \( s_r^{des} \) using (3.19), to guarantee the local stability of the whole state-space vector \( s \). Due to the physical constraints of the robots, we will reasonably assume that the angular velocity of the leader is bounded.
3.4. **PROPOSITION.** Let us suppose that $v_L > 0$, $|\omega_L| < W_{max}$, $|\beta(0)| < \pi$ and that $s_{r}^{des}$ is bounded. Then the control law (3.19)-(3.20) stabilizes the system dynamics (3.17)-(3.18).

**Proof:** Let us refer to $(e_{\rho}, e_{\psi})^T = s_r - s_{r}^{des}$ as the state tracking error vector. From (3.21), it follows that $(e_{\rho}, e_{\psi})^T$ is globally exponentially stable. We now prove that the internal dynamics is stable, i.e., that $|\beta|$ is bounded. Drawing $\omega_F$ from (3.19), equation (3.18) can be rewritten as:

$$\dot{\beta} = \frac{\sin \gamma}{d} (\dot{\rho}_{des} - k_1 e_{\rho}) - \frac{\rho \cos \gamma}{d} (\dot{\psi}_{des} - k_2 e_{\psi}) - \frac{v_L}{d} \sin \beta - \omega_L \left( \frac{\rho}{d} \cos \gamma - 1 \right).$$

(3.22)

Since $\omega_L$ and $s_{r}^{des}$ are bounded by hypothesis, then (3.7) can be re-written as:

$$\dot{\beta} = -\frac{v_L}{d} \sin \beta - B(t).$$

(3.23)

Note that without the term $B(t)$, a bounded persistent disturbance, (3.23) is locally asymptotically stable for $|\beta| < \pi$. From the stability theory of nonlinear systems with persistent disturbances (Slotine and Li 1991), being $|\beta(0)| < \pi$ and $B(t)$ bounded, it follows that $|\beta(t)| < \epsilon, \forall t > T$, for finite time $T$ and $\forall \epsilon > 0$.

3.5 Simulations and experimental results

3.5.1 Simulations

Simulations have been performed to test the validity of the observability analysis in Sect. 3.3. The following velocity input has been assigned to the leader,

$$v_L(t) = 0.3 \text{ m/s}$$

$$\omega_L(t) = \begin{cases} 0 \text{ rad/s} & \text{if } t \in \{0, 6, 14, 20, 28, 34\} \\ \pi/8 \text{ rad/s} & \text{otherwise} \end{cases}$$

which undergoes a piecewise rectilinear-circular trajectory, particularly suited for checking the necessity and sufficiency of the observability condition. The formation considered consists of two followers. We set $s(0) = (0.26, 2.183, 1.047, 0.368, 4.399, 0.524)^T$ and $s_{r}^{des} = (0.3, 3\pi/4, 0.3, 5\pi/4)^T$, where distances are in meters and angles in radians.

The gain matrix of the controller is $K = 6 I_4$. The EKF was initialized with $\hat{s}(0) - 1 = (\hat{\rho}_1(0), \hat{\psi}_1(0), \hat{\beta}_1(0), \hat{\rho}_2(0), \hat{\psi}_2(0), \hat{\beta}_2(0))^T$ corresponding to a 50% perturbation of the unknown distances to the leader and $P(0 - 1) = 10^{-2} \text{ diag}\{1, 1.1, 1.1, 1, 1.1, 1.1\}$. The other parameters are $T_c = 10 \text{ ms}$, $d = 0.1 \text{ m}$, $Q = \text{ diag}\{3 \times 10^{-5}, \varphi, \varphi, 3 \times 10^{-5}, \varphi, \varphi\}$ and $R = \varphi I_4$, where $\varphi = 0.9187 \times 10^{-4} \text{ rad}^2$. White Gaussian noise has been added to the measurements.
3.5. Simulations and experimental results

Figure 3.5: Simulation results: (a) Trajectory of the robots; (b) Time history of NEES and 95% bounds \((N = 15)\); (c) Time history of \(\det(J)\).

Fig. 3.5(a) shows the resulting trajectories of the three robots (in order to have a temporal reference the vehicles are drawn every two seconds). According to the results in Sect. 3.3.4, it is evident that the followers miss the formation exactly along the rectilinear tracts of the trajectory (e.g. \(t \in (14, 20]\)) where, since visual data are not changing sensibly, the system is not observable. On the other hand, when the leader switches from the rectilinear to the curvilinear tracts (e.g. \(t \in (20, 28]\)), a change in the visual information occurs, thus leading to an improvement of the localization: the desired
formation is in fact recovered soon after. In order to validate the above observations on more trials, we also studied the consistency of the EKF examining the time history of NEES, a concise representation of the estimation error (see the Appendix). Comparing Fig. 3.5(a) with Fig. 3.5(b) (here $r_1 = 4.38$, $r_2 = 7.87$ and $N = 15$), it is evident that the NEES tends to leave the 95% bounds exactly in correspondence of the rectilinear tracts of the leader trajectory (e.g. in $t \in (14, 20]$).

Fig. 3.5(c) reports the time history of $\det(\bar{J})$, with $\bar{J}$ is relative to follower 1 (similar results are obtained with $\bar{J}$ relative to follower 2). According to condition (3.13), we see that the state $s_1$ (analogously $s_2$) is observable along the curvilinear tracts of the trajectory, i.e. where $\det(\bar{J}) \neq 0$. Moreover, $\det(\bar{J})$ is near zero exactly along the straight tracts of the trajectory that also correspond to the time intervals in which the NEES increases. This confirms the observability results of Sect. 3.3.4, for which that the state $s_1$ (analogously $s_2$) is not observable.

3.5.2 Experiments

In order to validate the proposed formation control strategy in a real scenario, some experiments have been carried out at the GRASP Lab, University of Pennsylvania, Philadelphia. The experimental setup consists of two Scarab robots acting as a leader and a follower (see Fig. 3.6). Only one follower has been used in our experimental validation in order make the analysis of system’s observability as simple as possible. Actually, due to the special communication protocol adopted (cf. Sect. 3.2.3), we did not experience any significant performance change in the localization and formation control when multiple followers are used.
3.5. Simulations and experimental results

![Graph showing the trajectory of the leader and the follower. The time instants in which the leader switches from the rectilinear to the circular tracts and vice versa, are highlighted.](image)

**Figure 3.7**: Experimental results: (a) Trajectory of the leader and the follower. The time instants in which the leader switches from the rectilinear to the circular tracts and vice versa, are highlighted. (b)-(e) Snapshots from the experiment.
The Scarab is a differential driven robotic platform designed at the GRASP Lab, measuring $20 \text{ cm} \times 13.5 \text{ cm} \times 22.2 \text{ cm}$. The leader and the follower run identical modularized software with well-defined interfaces connecting modules via the Player robot architecture system. In order to provide a ground truth information of the actual robots pose, a tracking system consisting of LED markers on the top of each robot and eight overhead cameras are employed. More technical details on the Scarab robots and on the tracking system can be found in (Michael et al. 2008).

The robots are uniquely identified by colored markers and are equipped with a panoramic camera, consisting of a hyperbolic Remote Reality mirror (folded) screwed on a Point Grey Firefly IEEE 1394 camera. The image resolution is $320 \times 240$ pixels. Only the image principal point $(u_0, v_0)$ is known and is given by $(159.48, 123.70)$ pixels and $(172.89, 126.53)$ pixels, for the leader and the follower’s camera, respectively. HSV color blob detection is run on each robot using the Intel’s OpenCV libraries. Even though more sophisticated visual contour tracking algorithms are available in the literature (Blake and Isard 1998), with our blob detector we experienced a good compromise between real-time performances, robustness to changing in illumination, partial occlusions and tracking of far robots. The distance between the center of the robot and the marker is $d = 20 \text{ cm}$.

Note that, due to the presence of the LED marker on the top of each robot, the position of the panoramic camera on the vehicles is different from that shown in Fig. 3.1. However, a simple algebraic transformation is sufficient to readapt the robots’ angle measurement to the model in Fig. 3.1. This transformation has been implemented in the code running on the robots without significantly affecting the performance and the computational load.

For the experiment, we chose $s(0) = \dot{s}(0| - 1) = (0.75, \frac{5}{4\pi} , 0)^T$ and $s^\text{des} = (0.5, \frac{5}{4\pi})^T$. The control gains are $k_1 = k_2 = 0.5$ and $T_c = 0.1 \text{ s}$. Moreover, $P(0| - 1) = 10^{-2} I_3$, $Q = 10^{-5} \text{ diag}\{3, 9, 9\}$ and $R = 10^{-5} \text{ diag}\{9.1, 9.1\}$.

As in Sect. 3.5.1, we selected an input vector $(v_L, w_L)^T$ that gives rise to a rectilinear/circular trajectory.

Fig. 3.7(a) shows the trajectory of the robots during the whole experiment, from which we see that the follower is able to achieve the desired formation. The time instants in which the leader switches form the linear to the curvilinear trajectory and viceversa, have been highlighted in the figure. A series of snapshots from the experiment is reported in Figs. 3.7(b)-3.7(e). The range estimation error $\rho - \hat{\rho}$ and the range tracking error $\rho^\text{des} - \hat{\rho}$ are shown in Fig. 3.8(a) and 3.8(b), respectively. The errors on the view angles $\psi$ and $\beta$ are not reported here because, differently from the robot distance $\rho$, these components of the state vector are directly available in $y$ and are not critical to estimate.

\footnote{A video of the experiments is available at: www.dii.unisi.it/~gmariottini/MultiRobotLocalization09.m4v}
3.5. Simulations and experimental results

Figure 3.8: Experimental results: Time history of (a) the observation error $\rho - \hat{\rho}$; (b) the control error $\rho^{des} - \hat{\rho}$; (c) $\det(\bar{J})$.

As expected, the estimation and tracking errors decrease and stay close to zero approximately in $t \in [20, 55]$, corresponding to the circular tract of the trajectory. The value of $\det(\bar{J})$ in Fig. 3.8(c) is close to zero approximately at the same time instants in which both the tracking and estimation errors increase (e.g. for $t \in (55, 70]$), that is when the leader moves along the rectilinear tracts. This again confirms the observability results presented in Sects. 3.3.2 and 3.3.4.
3.6 Dealing with distant robots

Differently from what we have done so far, in this section we will assume that the only distinguishable feature detected by each panoramic camera, is the view-angle to the other observed robot’s center of gravity (or centroid). In practice, this corresponds to assume that the distance from the barycenter of the robot to the marker $P$ is zero, i.e., $d = 0$. This is a convenient choice (particularly suited, e.g., for airplane formations) because the centroid of a robot can be easily computed with standard computer vision techniques (Forsyth and Ponce 2002) also in the case of distant vehicles. On the other hand, a displaced marker on the body of the vehicle may be hard to detect even for near robots.

In the interest of brevity, we will not report simulation results in this section: the interested reader can find them in (Mariottini et al. 2007), where the closed-loop performance of the new feedback controller and the unscented Kalman filter (Julier and Uhlmann 2004) is studied. In the special case of $d = 0$, the leader-follower kinematic model (3.2) reduces to:

\[ \dot{s} = G(s)u, \]

where $s = (\rho, \psi, \beta)^T$, $u = (v_F, \omega_F, v_L, \omega_L)^T$ and,

\[ G(s) = \begin{bmatrix} \cos \gamma & 0 & -\cos \psi & 0 \\ -\frac{\sin \gamma}{\rho} & 0 & \frac{\sin \psi}{\rho} & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \]

being $\gamma \triangleq \beta + \psi$. As in Sect. 3.4, let us consider the set of kinematic equations equivalent to (3.24),

\[ \dot{s}_r = F(s)u_L + H(s)u_F \]

\[ \dot{\beta} = \omega_L - \omega_F, \]

where $s_r = (\rho, \psi)^T$ is a reduced state-space vector and as above $H(s)$ and $F(s)$ are the two upper left and right $2 \times 2$ submatrices of $G(s)$, respectively. The input-state feedback linearizing control proposed in Sect. 3.4 relies on the inversion of matrix $H(s)$, that turns out to be singular when $d = 0$. A possible solution to this problem is to design a feedback control via dynamic extension (Isidori 1995, Descusse and Moog 1985) with output vector $(\rho, \psi)^T$, by adding integrators on a subset of the input channels. A time differentiation of the first two equations in (3.24), yields,

\[ \begin{bmatrix} \ddot{\rho} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} -\dot{v}_L \cos \psi + v_L \dot{\psi} \sin \psi \\ \frac{\rho}{\dot{v}_L \sin \psi + v_L \dot{\psi} \cos \psi} \left( -v_L \dot{\rho} \sin \psi - \rho^2 \right) - \dot{\omega}_L \end{bmatrix} + \begin{bmatrix} \dot{v}_F \cos \gamma - v_F \dot{\gamma} \sin \gamma \\ -\frac{\rho}{\dot{v}_L \sin \gamma + v_F \dot{\gamma} \cos \gamma} - v_F \rho \sin \gamma \end{bmatrix}. \]
3.6. Dealing with distant robots

Substituting the first two lines of (3.24) in (3.27) and collecting the terms depending on the new input \( \mu \equiv (\dot{v}_F, \omega_F)^T \), we obtain,

\[
\begin{bmatrix}
\dot{\rho} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
\cos \gamma & v_F \sin \gamma \\
\sin \gamma & v_F \cos \gamma/\rho
\end{bmatrix} \begin{bmatrix}
\dot{v}_F \\
\omega_F
\end{bmatrix} + \begin{bmatrix}
\frac{1}{\rho} [v_L \sin \psi - v_F \sin \gamma] - \dot{v}_L \cos \psi - v_L \omega_L \sin \psi \\
\frac{2}{\rho^2} [v_F^2 \cos \gamma \sin \gamma + v_L^2 \cos \psi \sin \psi] - \omega_L - \frac{2 v_F v_L \sin(\gamma + \psi)}{\rho} + \frac{-v_L \omega_L \cos \psi + \dot{\psi} \sin \psi}{\rho}
\end{bmatrix} ,
\]

(3.28)

If \( v_F \neq 0 \), then \( C(s, v_F) \) represents a nonsingular counterpart of the decoupling matrix \( H(s) \), being \( \det(C(s, v_F)) = v_F/\rho \). Since \( C(s, v_F) \) is invertible, the following output-tracking control law can be implemented,

\[
\mu = C(s, v_F)^{-1} ( p - b(s, v_F)) ,
\]

(3.29)

where \( b(s, v_F) \) is the second term on the right-hand side of (3.28) and,

\[
p \equiv \begin{bmatrix}
\dot{\rho}^{des} + k_{21} (\dot{\rho}^{des} - \dot{\rho}) + k_{11} (\rho^{des} - \rho) \\
\dot{\psi}^{des} + k_{22} (\dot{\psi}^{des} - \dot{\psi}) + k_{12} (\psi^{des} - \psi)
\end{bmatrix} ,
\]

where \( \rho^{des} \) and \( \psi^{des} \) are the desired states. The polynomial \( z^2 + k_{2j} z + k_{1j} \), \( j = 1, 2 \), is Hurwitz, being \( k_{1j}, k_{2j} \in \mathbb{R} \) the controller gains. We conclude this section with two remarks:

i) In order to guarantee the closed loop stability, the study of the internal dynamics is required. In particular, due to the dynamic extension process, the extended state-space dimension is \( \tau = 4 \) and the system is described by,

\[
\begin{bmatrix}
\dot{\rho} \\
\dot{\psi} \\
\dot{\beta} \\
\dot{\psi}_F
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\dot{v}_F \\
\omega_F \\
\dot{\psi}_F
\end{bmatrix} + \begin{bmatrix}
v_F \cos \gamma - v_L \cos \psi \\
- v_F \sin \gamma + v_L \sin \psi \\
\rho \\
0
\end{bmatrix} ,
\]

(3.30)

However from (3.28), the total relative degree of (3.30) is 4 (for both the outputs \( \rho \) and \( \psi \)), that equals \( \tau \). Then, we can conclude that (3.30) has not internal dynamics and the output tracking control law (3.29) yields full-state linearization.
ii) In order to initialize the dynamic controller for an exact reproduction of the desired output trajectories, a common choice consists in assuming $\rho(0) = \rho^{\text{des}}(0)$ and $\psi(0) = \psi^{\text{des}}(0)$. Drawing $\tan \gamma$ from the first two lines of (3.24) and the leader’s known control inputs $v_L(0), \omega_L(0)$, we come up with\(^2\),

$$
\beta(0) = \text{atan2}\left\{ v_L(0) \sin(\psi(0)) - \rho(0)(\omega_L(0) + \dot{\psi}(0)), \; v_L(0) \cos(\psi(0)) + \dot{\rho}(0) \right\} - \psi(0).
$$

Analogously, we obtain,

$$
v_F(0) = \frac{\dot{\rho}(0) + v_L(0) \cos(\psi(0))}{\cos(\beta(0) + \psi(0))}.
$$

### 3.7 Observer via Immersion and Invariance

In (Astolfi and Ortega 2003), the authors presented a new methodology, called Immersion and Invariance (hereafter, I&I), to design asymptotically stabilizing and adaptive control laws for nonlinear systems. The method relies upon two classical tools from nonlinear regulator theory and geometric control: system immersion and manifold invariance. The I&I technique was subsequently adapted in (Karagiannis and Astolfi 2005) (see also (Karagiannis et al. 2008) for a more exhaustive presentation) to address the problem of constructing reduced-order observers for general nonlinear systems. In practice, the problem of designing a reduced-order observer is cast into the problem of rendering attractive an appropriately selected invariant manifold in the extended space of the plant and the observer. The effectiveness of the I&I observer design technique has been proved by Astolfi and coworkers through several academic and practical examples (see, e.g. (Carnevale et al. 2007, Carnevale and Astolfi 2008, Astolfi et al. 2008) and the references therein). However, this technique has never been applied in the multiagent literature up to today.

In Sect. 3.3.5, the extended Kalman filter has been used to estimate the range $\rho$. Although widely used in the literature, the EKF is known to have some drawbacks: it is difficult to tune and implement, the estimation error is not guaranteed to converge to zero and an a priori knowledge about noise is usually required. In order to overcome these drawbacks, in the rest of this section we will design a range estimator based on the I&I technique and study the closed loop stability of the system arising from the

\(^2\text{Recall that in the range } (-\pi, \pi),

\begin{align*}
\text{atan2}(y, x) & \triangleq \begin{cases} 
\arctan\left(\frac{y}{x}\right) \text{ sign } (y) & x > 0, \; y \neq 0 \\
\frac{\pi}{2} \text{ sign } (y) & x = 0, \; y \neq 0 \\
(\pi - \arctan\left(\frac{y}{x}\right)) \text{ sign } (y) & x < 0, \; y \neq 0
\end{cases} \\
\text{atan2}(0, x) & \triangleq \begin{cases} 
0 & x > 0 \\
\text{undefined} & x = 0 \\
\pi & x < 0
\end{cases}
\end{align*}
3.7. Observer via Immersion and Invariance

combination of the formation controller presented in Sect. 3.4 and the new estimator. The proposed reduced-order observer provides a globally asymptotically convergent estimate of the range, it can be easily tuned to achieve the desired convergence rate by acting on a single gain parameter and it is extremely simple to implement as well.

The leader-follower kinematic model we will consider in the sequel is exactly the same as in Sect. 3.2. However, in order to reduce ambiguity in the next derivations, a slightly different notation for the angles will be used (in blue, in Fig. 3.9). In this new notation, the leader-follower kinematics is given by,

\[
\begin{bmatrix}
\dot{\rho} \\
\dot{\psi} \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
\cos \gamma & d \sin \gamma & -\cos \psi & 0 \\
-\sin \gamma & d \cos \gamma & \sin \psi & -1 \\
0 & -1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
v_F \\
\omega_F \\
v_L \\
\omega_L
\end{bmatrix},
\]

(3.31)

where \( \gamma \equiv \phi + \psi \). As above, we suppose the leader measures two angles,

\[
y \equiv (y_1, y_2)^T = (\psi, \phi)^T,
\]

that will be used to estimate the range \( \rho \). We will assume that angles \( \zeta, \psi, \nu \in [0, 2\pi) \).

### 3.7.1 Observer design via I&I: an overview

For the reader’s convenience we provide here a brief overview of the basic theory related to the observer design via I&I (Karagiannis et al. 2008). Consider nonlinear, time-
3. Vision-based localization of robot formations

varying systems described by,

\[ \dot{y} = f_1(y, \eta, t) \]  \hspace{1cm} (3.33)

\[ \dot{\eta} = f_2(y, \eta, t) \]  \hspace{1cm} (3.34)

where \( y \in \mathbb{R}^m \) is the measured part of the state and \( \eta \in \mathbb{R}^q \) is the unmeasured part of the state. The vector fields \( f_1(\cdot), f_2(\cdot) \) are assumed to be forward complete, i.e. trajectories starting at time \( \bar{t} \) are defined for all times \( t \geq \bar{t} \).

3.3. DEFINITION. *The dynamical system,*

\[ \dot{\xi} = \alpha(y, \xi, t), \]  \hspace{1cm} (3.35)

with \( \xi \in \mathbb{R}^p, p \geq q \), is called an observer for the system (3.33)-(3.34), if there exist mappings,

\[ \beta(y, \xi, t) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \rightarrow \mathbb{R}^p, \quad \phi_{y,t}(\eta) : \mathbb{R}^q \rightarrow \mathbb{R}^p, \]

with \( \phi_{y,t}(\eta) \) parameterized by \( t \) and \( y \) and left-invertible \(^3\), such that the manifold,

\[ \mathcal{M} = \{ (y, \eta, \xi, t) \in \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R} : \beta(y, \xi, t) = \phi_{y,t}(\eta) \} , \]

has the following properties:

1. All trajectories of the extended system (3.33)-(3.35) that start on the manifold \( \mathcal{M} \) remain there for all future times, i.e. \( \mathcal{M} \) is positively invariant.

2. All trajectories of (3.33)-(3.35) that start in a neighborhood of \( \mathcal{M} \) asymptotically converge to \( \mathcal{M} \), i.e. \( \mathcal{M} \) is attractive.

The above definition states that an asymptotic estimate \( \hat{\eta} \) of \( \eta \) is given by \( \phi_{y,t}^L(\beta(y, \xi, t)) \), where \( \phi_{y,t}^L \) denotes a left-inverse of \( \phi_{y,t} \). The following proposition provides a general tool for constructing a nonlinear observer of the form given in Definition 3.3.

3.5. PROPOSITION. *Consider the system (3.33)-(3.35) and suppose that there exist two mappings* \( \beta(\cdot) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \rightarrow \mathbb{R}^p \) and \( \phi_{y,t}(\cdot) : \mathbb{R}^q \rightarrow \mathbb{R}^p \), *with a left-inverse* \( \phi_{y,t}^L(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^q \), *such that the following conditions hold:*

(A1) For all \( y, \xi \) and \( t \),

\[ \det \left( \frac{\partial \beta}{\partial \xi} \right) \neq 0. \]

\(^3\)A mapping \( \phi_{y,t}(\eta) : \mathbb{R}^q \rightarrow \mathbb{R}^p \) (parameterized by \( y \) and \( t \)) is *left-invertible* if there exists a mapping \( \phi_{y,t}^L(\eta) : \mathbb{R}^p \rightarrow \mathbb{R}^q \) such that \( \phi_{y,t}^L(\phi_{y,t}(\eta)) = \eta \), for all \( \eta \in \mathbb{R}^q \) (and for all \( y \) and \( t \)).
(A2) The system,
\[ \dot{z} = -\frac{\partial \beta}{\partial y} \left( f_1(y, \hat{\eta}, t) - f_1(y, \eta, t) \right) + \frac{\partial \phi_{y,t}}{\partial y} \bigg|_{\eta=\hat{\eta}} f_1(y, \hat{\eta}, t) - \frac{\partial \phi_{y,t}}{\partial y} f_1(y, \eta, t) \]
\[ + \frac{\partial \phi_{y,t}}{\partial \eta} \bigg|_{\eta=\hat{\eta}} f_2(y, \hat{\eta}, t) - \frac{\partial \phi_{y,t}}{\partial \eta} f_2(y, \eta, t) + \frac{\partial \phi_{y,t}}{\partial t} \bigg|_{\eta=\hat{\eta}} f_2(y, \hat{\eta}, t) - \frac{\partial \phi_{y,t}}{\partial t} f_2(y, \eta, t), \]
(3.36)

with \( \hat{\eta} = \phi_{y,t}^L(\beta(y, \xi, t)) \), has an asymptotically stable equilibrium at \( z = 0 \), uniformly in \( \eta, y \) and \( t \).

Then system (3.35) with,
\[ \alpha(y, \xi, t) = -\left( \frac{\partial \beta}{\partial \xi} \right)^{-1} \left( \frac{\partial \beta}{\partial y} f_1(y, \hat{\eta}, t) \right) + \frac{\partial \phi_{y,t}}{\partial y} \bigg|_{\eta=\hat{\eta}} f_1(y, \hat{\eta}, t) - \frac{\partial \phi_{y,t}}{\partial \eta} \bigg|_{\eta=\hat{\eta}} f_2(y, \hat{\eta}, t) - \frac{\partial \phi_{y,t}}{\partial \eta} f_2(y, \eta, t) - \frac{\partial \phi_{y,t}}{\partial t} \bigg|_{\eta=\hat{\eta}}, \]
where \( \hat{\eta} = \phi_{y,t}^L(\beta(y, \xi, t)) \), is a reduced-order observer for system (3.33)-(3.34).

3.2. Remark. Proposition 3.5 provides an implicit description of the observer dynamics (3.35) in terms of the mappings \( \beta(\cdot), \phi_{y,t}(\cdot) \) and \( \phi_{y,t}^L(\cdot) \) which must then be selected to satisfy (A2). Hence, the problem of constructing a reduced-order observer for the system (3.33)-(3.34) reduces to the problem of rendering the system (3.36) asymptotically stable by assigning the functions \( \beta(\cdot), \phi_{y,t}(\cdot) \) and \( \phi_{y,t}^L(\cdot) \). This peculiar stabilization problem can be extremely hard to solve, since, in general, it relies on the solution of a set of partial differential equations (or inequalities). However, as we will see in the next section, these equations are solvable in the problem under investigation.

3.7.2 I&I range estimator

In order to apply the procedure described in the previous section to design a nonlinear observer of the range \( \rho \), system (3.31) should be recast in the form (3.33)-(3.34). To this end, again, it is convenient to introduce the new variable \( \eta \equiv 1/\rho \), that is well-defined assuming \( \rho \neq 0 \). System (3.31) then becomes,
\[
\begin{bmatrix}
\dot{\eta} \\
\dot{\psi} \\
\dot{\phi}
\end{bmatrix} = \begin{bmatrix}
-\eta^2 \cos \gamma & -\eta^2 d \sin \gamma & \eta^2 \cos \psi & 0 \\
-\eta \sin \gamma & \eta d \cos \gamma & \eta \sin \psi & -1 \\
0 & -1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
v_F \\
\omega_F \\
v_L \\
\omega_L
\end{bmatrix},
\] (3.37)
Recalling that $y \triangleq (\psi, \varphi)^T$, system (3.37) can be rewritten as,

$$
\dot{y} = \begin{bmatrix}
-\omega_L & -v_F \sin \gamma + \omega_F \cos \gamma + \nu_L \sin y_1 \\
\omega_L - \omega_F & 0
\end{bmatrix} p(t) + \begin{bmatrix}
-\omega_L & -v_F \sin \gamma + \omega_F \cos \gamma + \nu_L \sin y_1 \\
\omega_L - \omega_F & 0
\end{bmatrix} g(y, t)
$$

$$
\dot{\eta} = - (v_F \cos \gamma + \omega_F \sin \gamma - \nu_L \cos y_1) \eta^2.
$$

The next proposition introduces a globally uniformly asymptotically convergent observer of $\eta$.

**3.6. PROPOSITION (RANGE ESTIMATOR).** Suppose that the control inputs of the robots are bounded functions of time, i.e. $v_L, \omega_L, v_F, \omega_F \in \mathcal{L}^\infty$ and that $v_L, v_F, \omega_F$ are first order differentiable. Suppose that the following condition holds,

$$
|g_1(y, t)| \geq \vartheta > 0,
$$

for some constant $\vartheta$ and for all $t$, where $g_1(y, t)$ is the first component of the vector $g(y, t)$ defined in (3.38). Then:

$$
\dot{\xi} = M (g_1(y, t), -v_F \sin \gamma + \omega_F \cos \gamma) (p(t) + g(y, t) \hat{\eta}) + M (\dot{v}_F \cos \gamma + \omega_F \sin \gamma - \dot{v}_L \cos y_1)
$$

$$
- \frac{\text{sign}(g_1(y, t))}{g_1(y, t)^2} \left( (\ell(y, t), v_F \cos \gamma + \omega_F \sin \gamma) (p(t) + g(y, t) \hat{\eta}) \hat{\eta}ight)
$$

$$
+ (\dot{v}_F \cos \gamma - \omega_F \sin \gamma - \dot{v}_L \sin y_1) \hat{\eta} + \frac{\text{sign}(g_1(y, t))}{g_1(y, t)} \ell(y, t) \hat{\eta}^2,
$$

is a globally uniformly asymptotically convergent observer for system (3.38), where $M$ is a positive gain to be suitably tuned and,

$$
\hat{\eta} = (M \ell(y, t) - \xi) |g_1(y, t)|.
$$

**Proof:** With reference to the general design procedure presented in Sect. 3.7.1, let us suppose for simplicity, that,

$$
\phi_{y,t}(\eta) = \varepsilon(y, t) \eta,
$$

where $\varepsilon(\cdot) \neq 0$ is a function to be determined (Carnevale et al. 2007). Consider an observer of the form given in Proposition 3.5,

$$
\dot{\xi} = - \left( \frac{\partial \beta}{\partial \xi} \right)^{-1} \left( \frac{\partial \beta}{\partial y} (p(t) + g(y, t) \hat{\eta}) + \frac{\partial \beta}{\partial t} - \frac{\partial \varepsilon}{\partial y} (p(t) + g(y, t) \hat{\eta}) \hat{\eta} - \frac{\partial \varepsilon}{\partial t} \hat{\eta} + \varepsilon(y, t) \ell(y, t) \hat{\eta}^2 \right),
$$

$$
\dot{\eta} = \varepsilon(y, t)^{-1} \beta(y, \xi, t).
$$

(3.42)
3.7. Observer via Immersion and Invariance

From (3.36) the dynamics of the error,

\[ z = \beta(y, \xi, t) - \varepsilon(y, t) \eta = \varepsilon(y, t)(\dot{\eta} - \eta), \]

is given by,

\[ \dot{z} = -\left( \frac{\partial \beta}{\partial y} g(y, t) - \frac{\partial \varepsilon}{\partial y} p(t) - \frac{\partial \varepsilon}{\partial t} \right) \varepsilon(y, t)^{-1} z + \left( \frac{\partial \varepsilon}{\partial y} g(y, t) - \varepsilon(y, t) \ell(y, t) \right)(\dot{\eta}^2 - \eta^2). \] (3.43)

The observer design problem is now reduced to finding functions \( \beta(\cdot) \) and \( \varepsilon(\cdot) \neq 0 \) that satisfy assumptions (A1)-(A2) of Proposition 3.5. In view of (3.43) this can be achieved by solving the partial differential equations,

\[ \frac{\partial \beta}{\partial y} g(y, t) - \frac{\partial \varepsilon}{\partial y} p(t) - \frac{\partial \varepsilon}{\partial t} = \kappa(y, t) \varepsilon(y, t), \] (3.44)

\[ \frac{\partial \varepsilon}{\partial y} g(y, t) - \varepsilon(y, t) \ell(y, t) = 0, \] (3.45)

for some \( \kappa(\cdot) > 0 \). From (3.45) we obtain the solution,

\[ \varepsilon(y, t) = -|g_1(y, t)|^{-1}, \]

which by (3.39) is well-defined and nonzero for all \( y \) and \( t \). Let

\[ \kappa(y, t) = M|g_1(y, t)|^3 + \left( \frac{\partial \varepsilon}{\partial y} p(t) + \frac{\partial \varepsilon}{\partial t} \right)|g_1(y, t)|. \]

By boundedness of the control inputs and \( y(t) \), it is always possible to find \( M > 0 \) (sufficiently large) such that \( \kappa(\cdot) > 0 \). Equation (3.44) is now reduced to,

\[ \frac{\partial \beta}{\partial y} g(y, t) = -Mg_1(y, t)^2, \]

which can be solved for \( \beta(\cdot) \) yielding,

\[ \beta(y, \xi, t) = -M\ell(y, t) + \tau(\xi, t), \]

where \( \tau(\cdot) \) is a free function. Selecting \( \tau(\xi, t) = \xi \) ensures that assumption (A1) is satisfied. Substituting the above expression into (3.43) yields the equation \( \dot{z} = -\kappa(y, t)z \) which has uniformly asymptotically stable equilibrium at zero, hence assumption (A2) holds. By substituting the expressions of \( \varepsilon(\cdot) \) and \( \beta(\cdot) \) (with \( \tau(\xi, t) = \xi \)) in (3.42), we obtain (3.40)-(3.41).
3. Vision-based localization of robot formations

Some observations are in order at this point:

- Equation (3.40) is a reduced-order observer for system (3.38): in fact it has lower dimension than the system.
- The observer (3.40) can be easily tuned to achieve the desired convergence rate by acting on the single gain parameter $M$.

3.3. REMARK. Note that (3.39), which is necessary to avoid singularities in (3.40), exactly corresponds to the observability condition derived in Sect. 3.3.2 by studying the singularity of the Extended Output Jacobian.

3.7.3 Formation control and closed-loop stability

Note that if the state $s = (\eta, \psi, \varphi)^T$ was perfectly known, then system (3.37) could be exactly input-state feedback linearized and the asymptotic convergence of $s$ towards a desired state $s^{des}$ guaranteed. The presence of an observer inside the control loop obviously makes the convergence analysis more involved. In Proposition 3.7, we design a formation controller and study the effect of the I&I observer on the closed-loop stability by means of a Lyapunov analysis. To this end, it is convenient to rewrite system (3.37) in the form:

$$
\dot{s}_r = F(s) u_L + H(s) u_F \tag{3.46}
$$

$$
\dot{\varphi} = \omega_L - \omega_F \tag{3.47}
$$

where $s_r = (\eta, \psi)^T$, as usual, is the reduced state space vector, $u_L = (v_L, \omega_L)^T$, $u_F = (v_F, \omega_F)^T$ and,

$$
F(s) = \begin{bmatrix} \eta^2 \cos \psi & 0 \\ \eta \sin \psi & -1 \end{bmatrix}, \quad H(s) = \begin{bmatrix} -\eta^2 \cos \gamma & -\eta^2 d \sin \gamma \\ -\eta \sin \gamma & \eta d \cos \gamma \end{bmatrix}.
$$

3.7. PROPOSITION (CONTROL AND CLOSED-LOOP STABILITY). Consider system (3.46)-(3.47) and suppose that $v_L > 0$ and $|\omega_L| \leq \omega_{L_{\max}}$, $\omega_{L_{\max}} > 0$. For a given state estimate $\hat{s} = (\hat{\eta}, \psi, \varphi)^T$ (with $\hat{\eta} > 0$) provided by the observer in Proposition 3.6 with gain $M$ sufficiently large, the feedback control law,

$$
u_F = H^{-1}(\hat{s}) (p - F(\hat{s}) u_L), \tag{3.48}
$$

with $p \triangleq -K(\hat{s}_r - s^{des}_r)$, $K = diag\{k_1, k_2\}$, $k_1, k_2 > 0$, $\hat{s}_r = (\hat{\eta}, \psi)^T$, guarantees the asymptotic convergence of the control error $s_r - s^{des}_r$ to zero and the locally uniformly ultimate boundedness (UUB) of the internal dynamics $\varphi$. 

Proof: Substituting (3.48) in (3.46) we obtain the dynamics of the controlled system \( \dot{s}_r = F(s)u_L + H(s)H^{-1}(s)(p(s_r) - F(s)u_L) \). Since \( s_r^{des} \) is constant, the dynamics of the control error \( e_r = s_r - s_r^{des} \) is,

\[
\dot{e}_r = \begin{bmatrix}
-k_1(\eta/\hat{\eta})^2 & 0 \\
0 & -k_2(\eta/\hat{\eta})
\end{bmatrix} e_r + \begin{bmatrix}
-k_1(\eta - \eta)(\eta/\hat{\eta})^2 \\
\omega_L(\eta/\hat{\eta} - 1)
\end{bmatrix},
\tag{3.49}
\]

where \( \dot{s}_r = s_r + (\hat{\eta} - \eta, 0)^T \). To prove that the control error asymptotically converges to zero, we should study the stability of a linear time-varying system with perturbation \( b(t) \). Let us first study the stability of the equilibrium point \( e_r = 0 \) of the non-perturbed system. Given the candidate Lyapunov function \( V = e_r^T e_r \), we have,

\[
\dot{V} = e_r^T \dot{e}_r + \dot{e}_r^T e_r = 2 e_r^T A(t) e_r \leq 2 \lambda_M \|e_r\|^2 = 2 \lambda_M V,
\]

where \( \lambda_M = \max\{-k_1(\eta/\hat{\eta})^2, -k_2(\eta/\hat{\eta})\} \). Since \( \hat{\eta} > 0 \), then \( \lambda_M < 0 \), which implies that \( e_r = 0 \) is a globally asymptotically stable equilibrium point for the non-perturbed system. To study the stability of the perturbed system, let us consider again the Lyapunov function \( V = e_r^T e_r \) for which it results:

\[
\dot{V} = 2 e_r^T A(t) e_r + 2 e_r^T b(t) \leq 2 \lambda_M \|e_r\|^2 + 2 \|e_r\| \|b(t)\|
\leq 2 (1 - \theta) \lambda_M \|e_r\|^2 + 2 \lambda_M \|e_r\|^2 + 2 \|e_r\| \delta,
\tag{3.50}
\]

where \( 0 < \theta < 1 \) and \( \|b(t)\| \leq \delta \). From the last inequality in (3.50) we have,

\[
\dot{V} \leq 2 (1 - \theta) \lambda_M \|e_r\|^2 < 0 \quad \text{if} \quad \delta \leq - \theta \lambda_M \|e_r\|, \quad \forall \, e_r.
\]

Since \( |\omega_L| \leq \omega_{L_{max}} \), we can choose,

\[
\delta = \left| \frac{\eta}{\hat{\eta}} - 1 \right| \sqrt{\omega_{L_{max}}^2 + k_1^2 \frac{\eta^4}{\hat{\eta}^2}},
\]

and rewrite \( \delta \leq - \theta \lambda_M \|e_r\| \) as:

\[
\|e_r\| \geq - \frac{1}{\theta \lambda_M} \left| \frac{\eta}{\hat{\eta}} - 1 \right| \sqrt{\omega_{L_{max}}^2 + k_1^2 \frac{\eta^4}{\hat{\eta}^2}}. \tag{3.51}
\]

We now study under which conditions (3.51) is verified, that is, \( e_r = 0 \) is an asymptotically stable equilibrium point for the perturbed system. If \( \hat{\eta} \) rapidly converges to \( \eta \), we note that inequality (3.51) reduces to \( \|e_r\| \geq 0 \), that is always true. This implies that \( e_r = 0 \) is an asymptotically stable equilibrium point for system (3.49). Note that due to the exponential
convergence of the I&I observer estimation error to zero, there will exist two positive constants $D$ and $C$ such that,

$$\hat{\eta} \geq D e^{-Ct} + \eta,$$

or equivalently,

$$\left| 1 - \frac{\hat{\eta}}{\eta} \right| \geq \frac{D}{\eta} e^{-Ct}.$$

Using this inequality in (3.51) and observing that parameter $C$ is proportional to the gain $M$, we see that the asymptotic convergence of the control error to zero can be always guaranteed by choosing $M$ sufficiently large. It now remains to show that the internal dynamics $\varphi$ is locally UUB. Exploiting $\omega_F$ from (3.48), we can rewrite equation (3.47) as,

$$\dot{\varphi} = -\frac{v_L}{d} \sin \varphi - \frac{\sin \gamma}{\eta^2 d} k_1 e_r(1) + \frac{\cos \gamma}{\eta d} k_2 e_r(2) - \omega_L \left( \frac{\cos \gamma}{\eta d} - 1 \right),$$

or more synthetically as,

$$\dot{\varphi} = -\frac{v_L}{d} \sin \varphi + B(t, \varphi), \quad (3.52)$$

where $B(t, \varphi)$ is a nonvanishing perturbation acting on the nominal system $\dot{\varphi} = -\frac{v_L}{d} \sin \varphi$. The nominal system has a locally uniformly asymptotically stable equilibrium point in $\varphi = 0$ and its Lyapunov function $V = \frac{1}{2} \varphi^2$ satisfies the inequalities (Khalil 2002):

$$\alpha_1(|\varphi|) \leq V \leq \alpha_2(|\varphi|),$$

$$-\frac{\partial V}{\partial \varphi} \frac{v_L}{d} \sin \varphi \leq -\alpha_3(|\varphi|),$$

$$\left| \frac{\partial V}{\partial \varphi} \right| \leq \alpha_4(|\varphi|),$$

in $[0, \infty) \times \mathcal{G}$, where $\mathcal{G} = \{ \varphi \in \mathbb{R} : |\varphi| < \epsilon \}$, being $\epsilon$ a sufficiently small positive constant. $\alpha_i(\cdot)$, $i = 1, \ldots, 4$, are class $\mathcal{K}$ functions defined as follows: $\alpha_1 = \frac{1}{4} \varphi^2$, $\alpha_2 = \varphi^2$, $\alpha_3 = \frac{v_L}{d} \varphi^2$ and $\alpha_4 = 2 |\varphi|$. Since $e_r$ is asymptotically convergent to zero and, by hypothesis $\omega_L$ is bounded, there exist suitable velocities for the leader such that $B(t, \varphi)$ satisfies the uniform bound,

$$|B(t, \varphi)| \leq \delta < \frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(\epsilon)))}{\alpha_4(\epsilon)} \leq \frac{v_L \theta \epsilon}{8 d},$$

for all $t \geq 0$, all $\varphi \in \mathcal{G}$ and $0 < \theta < 1$. Then, for all $\varphi(0) < \alpha_2^{-1}(\alpha_1(\epsilon)) = \epsilon / 2$.

---

A continuous function $\sigma : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$. 

---
3.8. Conclusions and future work

the solution $\varphi(t)$ of the perturbed system (3.52), satisfies $|\varphi(t)| \leq \chi(|\varphi(0)|, t)$, for all $0 \leq t < t_1$ and $|\varphi(t)| \leq \sigma(\delta)$, $\forall t \geq t_1$ for some class $\mathcal{KL}$ function\footnote{A continuous function $\chi : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed $j$, the mapping $\chi(i, j)$ belongs to class $\mathcal{K}$ with respect to $i$ and, for each fixed $i$, the mapping $\chi(i, j)$ is decreasing with respect to $j$ and $\chi(i, j) \rightarrow 0$ as $j \rightarrow \infty$.} $\chi(\cdot, \cdot)$ and some finite time $t_1$, where $\sigma(\delta)$ is a class $\mathcal{K}$ function of $\delta$ defined as: $\sigma(\delta) = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(\frac{\delta \alpha(c)}{\theta})))) \triangleq 2 \sqrt{\frac{2d \delta \epsilon}{\theta v L}}$. This proves that $\varphi(t)$ is locally UUB.

3.7.4 Simulation results

Simulation experiments have been conducted to study the closed-loop performance of the observer via I&I and the formation controller. The leader is supposed to move along a circular path with velocities $v_L = 1$ m/s and $\omega_L = \pi/10$ rad/s. The initial condition of system (3.37) is $(\eta(0), \psi(0), \varphi(0))^T = (0.7186, 1.5013, 0.2618)^T$, the desired state $s_{des}^T = (1, 2\pi/3)^T$ and $d = 0.1$ m. The gains of the controller and observer are $k_1 = k_2 = 0.1$ and $M = 12$, respectively. These values, as requested in Proposition 3.7, guarantee that the convergence rate of the observer is faster than that of the controller. Fig. 3.10(a) shows the trajectory of the leader and the follower (to provide a time reference, the robots are drawn every two seconds). Fig. 3.10(b) reports the time history of the observation error $\rho - \hat{\rho}$. The error asymptotically converges to zero as expected. In Fig. 3.10(c) the control errors $\rho - \rho_{des}$ and $\psi - \psi_{des}$ asymptotically converge to zero (recall that $\rho \triangleq 1/\eta$). Fig. 3.10(d) reports the control inputs $v_F$ and $\omega_F$ and Fig. 3.10(e) the time history of the bearing angle $\varphi$. In accordance with Proposition 3.7, the internal dynamics $\varphi$ remains bounded while the desired formation is achieved.

From the simulation experiments, we have noticed that a guideline for the gains is to select $M$ from 1 to 2 orders of magnitude greater than $k_1$, $k_2$ and that the size of $M$ is not affected by the sampling time chosen to numerically integrate equation (3.40). Fig. 3.11 shows the performance of the observer via I&I and the feedback controller for $k_1 = k_2 = 0.1$ and $M$ variable. Fig. 3.11(a) reports the time history of the observation error $\rho - \hat{\rho}$ and Fig. 3.11(b) the control errors $\rho - \rho_{des}$ and $\psi - \psi_{des}$.

3.8 Conclusions and future work

In this chapter we have studied the vision-based localization and control of leader-follower formations of nonholonomic mobile robots. Each robot is equipped with a panoramic camera which only provides the view-angle to the other vehicles. As an original contribution, the localization problem has been addressed using a new observability condition based on the Extended Output Jacobian. The state of the leader-follower
3. Vision-based localization of robot formations

![Figure 3.10](image)

Figure 3.10: (a) Trajectory of the leader and the follower; (b) Observation error $\rho - \hat{\rho}$; (c) Control errors $\rho - \rho^{\text{des}}$ and $\psi - \psi^{\text{des}}$; (d) Control inputs $v_F$ and $\omega_F$; (e) Bearing angle $\varphi$. 
system is estimated via the extended Kalman filter and an input-state feedback control law is designed to stabilize the formation. A more general working scenario is explored at the end of the chapter and a new range estimator based on the Immersion and Invariance technique is presented. Simulations as well as real-data experiments performed with Scarab robots, illustrate the theory and show the effectiveness of the proposed designs.

Future research directions include the study of a decentralized version of our formation control strategy and the integration of different sensor typologies, such as, e.g., laser range finders and inertial measurement units (IMUs).

Appendix: consistency of a state estimator

3.4. Definition. A state estimator is said to be consistent (Bar-Shalom et al. 2001) if its state estimation error \( \hat{s}(k|k) \triangleq \hat{s}(k) - s(k|k) \) is such that,

\[
E \left[ \hat{s}(k|k) \right] = 0, \quad E \left[ \hat{s}(k|k) \hat{s}(k|k)^T \right] = P(k|k).
\]

To practically evaluate the consistency of an estimator, the normalized estimation error squared (NEES) is defined:

\[
\varepsilon(k) \triangleq \hat{s}(k|k)^T P^{-1}(k|k) \hat{s}(k|k).
\]
Let us consider $N$ Monte Carlo simulations that provide $N$ samples $\varepsilon_i(k)$ of the random variable $\varepsilon(k)$. Let

$$\bar{\varepsilon}(k) = \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i(k),$$

be the sample mean of $\varepsilon(k)$. The hypothesis that the state estimation errors are consistent with the estimator calculated covariances is not invalidated if,

$$\bar{\varepsilon}(k) \in [r_1, r_2], \ r_1, r_2 \in \mathbb{R}.$$

Under the Gaussian assumption, $N\bar{\varepsilon}(k) \sim \chi^2_{N3q}$ where $\chi^2_{N3q}$ is a $N \times 3q$ degrees of freedom Chi-square distribution. $r_1, r_2$ can then be computed from a table providing the points on the Chi-square distribution for a given tail probability (see, e.g. (Bar-Shalom et al. 2001, Appendix C)). It is worth recalling that even if specifically designed for linear systems, the consistency criterion based on the NEES is commonly used also in the nonlinear case.
Chapter 4

Visibility maintenance via controlled invariance

Science cannot solve the ultimate mystery of nature. And that is because, in the last analysis, we ourselves are a part of the mystery that we are trying to solve.

M. Planck

Abstract

In this chapter we study the visibility maintenance problem (VMP) for a leader-follower pair of Dubins-like vehicles with input constraints and propose an original solution based on the notion of controlled invariance. The nonlinear model describing the relative dynamics of the vehicles is interpreted as linear uncertain system, with the leader robot acting as an external disturbance. The VMP can then be reformulated as a linear constrained regulation problem with additive disturbances (DLCRP). New positive D-invariance conditions for linear uncertain systems with parametric disturbance matrix are introduced and used to solve the VMP when box bounds on the state, control input and disturbance are considered. The proposed design procedure can be easily adapted to provide the control with UBB disturbances rejection capabilities and extended to more general scenarios. Simulation experiments show the applicability of the proposed approach.

The material of this chapter is drawn from (Morbidi, Bullo and Prattichizzo 2008).

4.1 Introduction

In order to make the analysis of multiagent systems more accurate and realistic, additional constraints have been gradually incorporated in the basic model of a robotic network in the last few years. It has been assumed, for instance, that the measurements are quantized and/or noisy, that random delays are present in the communication, that the communication channel is packet-lossy, that the connectivity range is limited. If multiple robots with limited communication capabilities (usually modeled with disks of finite radii centered on the robots) have to perform complex tasks, it is generally hard to guarantee the connectivity is preserved all the time. This is usually referred to as the connectivity maintenance problem in the literature and it stimulated a

If the robots are equipped with sensors (such as, e.g. panoramic cameras, laser range finders, high resolution radars, etc.) having limited sensing capabilities, then a similar problem, called visibility maintenance problem (VMP), comes into play.

Visibility constraints have been introduced in several works dealing with pursuit-evasion (Gerkey et al. 2006, Tovar and LaValle 2008), collective circular motion (Ceccarelli et al. 2008), deployment (Ganguli et al. 2007) and rendezvous (Lin et al. 2004, Ganguli et al. 2008, Conditionally accepted). However, these works do not study how to actively enforce visibility among the robots via a suitable control action and visibility never really becomes a system specification therein. In addition, bounds on the velocities of the robots have not been explicitly taken into account in most of these works.

The setup considered in this chapter consists of two agents with first-order dynamics: a leader (or evader) L and a follower (or pursuer) F. The robots can rotate, but similarly to Dubins’ vehicles can only move forward. The follower is equipped with a sensing device characterized by a visibility set $S$, a compact and convex polyhedral region embedding both position and angle information. The leader moves along an arbitrary trajectory: the aim of the follower is to maintain L always inside its visibility set $S$, while respecting suitable bounds on the control inputs.

Inspired by (Tiwari et al. 2004), where the concept of cone invariance is employed to solve the multiagent rendezvous problem and by the results in (Blanchini 1990, Blanchini 1991), we will address the VMP using the notion of controlled invariance. The key idea is to interpret the nonlinear model describing the relative dynamics of the leader and the follower, as a linear system with model parameter uncertainty, with the control input of the leader acting as an external disturbance. The VMP can then be easily reformulated as linear constrained regulation problem with additive disturbances (DLCRP) (Blanchini 1991). New positive $D$-invariance conditions for general linear uncertain systems with parametric disturbance matrix are introduced and used to solve the VMP when box bounds on the visibility set, control inputs and disturbances are considered. Analytical conditions for the solution of the VMP are obtained by symbolically solving the set of linear inequalities defining the polytope of all the feasible state feedback matrices, using the Fourier-Motzkin elimination method. The proposed design procedure can be easily adapted to provide the control with UBB disturbances rejection capabilities. Some other extensions are discussed at the end of the chapter: we present conditions for the solution of the VMP on a circle and extend the results valid for a leader-follower pair of robots to a chain of $n$ vehicles. Simulation results are provided to illustrate the theory.
4.2. The linear constrained regulation problem

The rest of the chapter is organized as follows. In Sect. 4.2 the linear constrained regulation problem is reviewed and new positive $D$-invariance conditions for linear systems with parameter uncertainty are presented. In Sect. 4.3 we introduce the VMP and prove the main results of the chapter. In Sect. 4.4 the results of simulation experiments are discussed. In Sect. 4.5 the main contributions of the chapter are summarized and future research lines are highlighted. The appendix provides a brief overview of the Fourier-Motzkin elimination method.

4.2 The linear constrained regulation problem

This section presents a series of results that are instrumental to address the VMP in Sect. 4.3. Theorem 4.6, Corollary 4.1, Theorem 4.7 and Corollary 4.2 extend the analogous results in (Blanchini 1990, Blanchini 1991) to linear uncertain systems with parameter disturbance matrix and are an original contributions of this work. Consider the following system,

$$
\dot{s}(t) = A(q(t)) s(t) + B(q(t)) u(t),
$$

where $s(t) \in \mathcal{X} \subset \mathbb{R}^n$ and $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ are respectively the state and input vectors, $q(t) \in \mathcal{Q} \subset \mathbb{R}^p$ is the model parameter uncertainty vector, while $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Q}$ are assigned sets containing the origin, with $\mathcal{U}$ and $\mathcal{Q}$ compact. We assume that $A(q)$ and $B(q)$ are matrices of suitable dimensions whose entries are continuous functions of $q$. We shall assume $q(t)$ to be a piecewise continuous function of time.

4.1. Definition (Positive invariance). The set $S \subset \mathbb{R}^n$ is positively invariant for system (4.1), if and only if, for every initial condition $s(0) \in S$ and every admissible $q(t) \in \mathcal{Q}$, the solution obtained for $u(t) \equiv 0$, satisfies the condition $s(t) \in S$ for $t > 0$.

4.2. Definition (Admissible region). A region $S \subset \mathbb{R}^n$ is said to be admissible for the feedback control law $u = Ks$, if and only if, for every $s \in S$, the condition $u \in \mathcal{U}$ holds. If $\mathcal{U}$ and $S$ are convex polyhedral sets containing the origin, the admissibility of $S$ is simply equivalent to,

$$
K v_i \in \mathcal{U}, \quad v_i \in \text{vert}(S), \quad i = 1, \ldots, \mu,
$$

where $\text{vert}(S)$ denotes the set of vertices of $S$.

We can now introduce the linear constrained regulation problem (LCRP), (Blanchini 1991).

4.1. Problem (LCRP). Given a system in the form (4.1), find a linear feedback control law $u(t) = Ks(t)$ and a set $S \subset \mathcal{X}$ such that, for every initial condition $s(0) \in S$ and every admissible function $q(t) \in \mathcal{Q}$, the conditions $s(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ are fulfilled for $t > 0$. 
4.1. **Theorem.** The LCRP has a solution if and only if there exists a feedback matrix $K$ and a set $S \subset \mathcal{X}$ that is positive invariant and admissible for the closed loop system,

\[
\dot{s}(t) = F(q(t)) s(t),
\]

where $F(q(t)) = A(q(t)) + B(q(t))K$.

The notion of tangent cone (Blanchini 1999) will be useful in the sequel.

4.3. **Definition (Tangent Cone).** Let $S \subset \mathbb{R}^n$ be a compact and convex set. If $s_0 \in \partial S$, we define the following set as the tangent cone (or contingent cone) to $S$ at $s_0$:

\[
T_S(s_0) = \left\{ z \in \mathbb{R}^n : \liminf_{t \to 0^+} \frac{\text{dist}(s_0 + tz, S)}{t} = 0 \right\},
\]

where $\text{dist}(s_0, S) = \inf_{x \in S} \|s_0 - x\|_2$.

4.2. **Theorem (Sub-tangentiality Condition).** Let $S \subset \mathbb{R}^n$ be a compact and convex set with nonempty interior. The positive invariance of $S$ for (4.1) is equivalent to the following condition: for every $s_0 \in \partial S$ and $q \in \mathcal{Q}$,

\[
A(q) s_0 \in T_S(s_0),
\]

where $T_S(s_0)$ is the tangent cone to $S$ at $s_0$ (Blanchini 1999).

The main difficulty in exploiting condition (4.4) to study the positive invariance of an assigned region $S$ is that it has to be checked on the boundary of $S$. However, if convex polyhedral sets are considered, only their vertices must be taken into account and easy algebraic conditions can be derived. In this respect, let us consider a system of the form (4.1), with,

\[
A(q(t)) = A_0 + \sum_{l=1}^{p} A_l q_l(t), \quad B(q(t)) = B_0 + \sum_{l=1}^{p} B_l q_l(t),
\]

where $A_l$ and $B_l$, $l = 1, \ldots, p$, are constant matrices of appropriate dimension and $q(t)$ takes values in a compact and convex polyhedron $\mathcal{Q} \subset \mathbb{R}^p$. Let the set $\mathcal{U}$ be compact, convex and polyhedral as well. We consider a candidate convex and compact polyhedral set $S$ containing the origin in its interior and we search for a feedback matrix $K$ that assures the positive invariance of $S$ for the closed loop system (4.3). Since $S$ is polyhedral, then condition (4.4) is fulfilled on $\partial S$ if and only if is fulfilled on every vertex of $S$.

4.3. **Theorem.** The set $S$ is positive invariant for system (4.3) with feedback $u = Ks$, if and only if, for all $v_i \in \text{vert}(S)$ and $w_j \in \text{vert}(\mathcal{Q})$:
4.2. The linear constrained regulation problem

\[ F(w_j) v_i \in T_S(v_i), \quad i = 1, \ldots, \mu, j = 1, \ldots, \nu. \]

The LCRP as formulated in Problem 4.1 does not require the stability. However, a desirable property is the global uniform stability of the closed loop system. The relationship between the stability property and the existence of positively invariant regions is established by next theorem (Blanchini 1991).

4.4. Theorem. The system \( \dot{s}(t) = F(q(t)) s(t) \) where \( F \) depends continuously on \( q(t) \in Q, t \geq 0 \), with \( Q \) compact and \( q(t) \) piecewise continuous, is uniformly stable if and only if there exists a convex and compact region \( S \) containing the origin in its interior, which is positively invariant for the system.

We see that if we find a compact and convex set \( S \) (containing the origin in its interior) positively invariant for the closed loop system, then the stability is achieved. The asymptotic stability is assured if the region \( S \) has the following property: for all \( s(0) \in \partial S \) and every admissible \( q(t) \), the solution \( s(t) \) belongs to the interior of \( S \) for \( t > 0 \). If polyhedral sets are considered, to assure this condition we may replace the inequalities arising from Theorem 4.3 with strict inequalities.

Let us now turn our attention to systems in the form,

\[ \dot{s}(t) = A(q(t)) s(t) + B(q(t)) u(t) + E(q(t)) \delta(t), \quad (4.5) \]

where the unknown external disturbance \( \delta(t) \) is constrained in a compact and convex polyhedral set \( D \subset \mathbb{R}^l \) containing the origin. Note that with respect to the systems considered in (Blanchini 1991), the structure of (4.5) is more general since matrix \( E \) also depends on the uncertain parameter \( q \). As an immediate extension of the positive invariance property introduced in Definition 4.1, we may require that the state \( s \) remains in \( S \) despite the presence of the disturbance \( \delta(t) \).

4.4. Definition (Positive \( D \)-invariance). The set \( S \subset \mathbb{R}^n \) is positively \( D \)-invariant (PDI) for system (4.5), if for every initial condition \( s(0) \in S \) and all admissible \( q(t) \in Q \) and \( \delta(t) \in D \), the solution obtained for \( u(t) \equiv 0 \), satisfies the condition \( s(t) \in S \) for \( t > 0 \).

We can now introduce the linear constrained regulation problem with additive disturbances (DLCRP).

4.2. Problem (DLCRP). Given a system in the form (4.5), find a linear feedback control law \( u(t) = K s(t) \) and a set \( S \subset X \) such that, for every initial condition \( s(0) \in S \) and every admissible \( q(t) \in Q \) and \( \delta(t) \in D \), the conditions \( s(t) \in X \) and \( u(t) \in U \) are fulfilled for \( t > 0 \).

4.5. Theorem. The DLCRP has a solution if and only if there exists a feedback matrix \( K \) and a set \( S \subset X \) that is PDI and admissible for the closed loop system \( \dot{s}(t) = F(q(t)) s(t) + E(q(t)) \delta(t) \).
Similarly to $A(q(t))$ and $B(q(t))$, we will henceforth suppose that,

$$E(q(t)) = E_0 + \sum_{l=1}^{p} E_l q_l(t).$$

4.6. Theorem. The set $S$ is positively $\mathcal{D}$-invariant for system (4.5) with feedback $u = Ks$, if and only if, for all $v_i \in \text{vert}(S)$, $\omega_j \in \text{vert}(Q)$ and $r_k \in \text{vert}(\mathcal{D})$,

$$F(w_j) v_i + E(w_j) r_k \in T_S(v_i), \quad i = 1, \ldots, \mu, \ j = 1, \ldots, \nu, \ k = 1, \ldots, \eta. \quad (4.6)$$

Proof: For the necessity, we have to prove that if $S$ is a positive $\mathcal{D}$-invariant region for system (4.5), then condition (4.6) holds. The proof is straightforward, since for the sub-tangentiality condition the positive $\mathcal{D}$-invariance of $S$ for system (4.5) is equivalent to,

$$F(q) s + E(q) \delta \in T_S(s), \ s \in \partial S, \ q \in Q, \ \delta \in \mathcal{D}, \quad (4.7)$$

that trivially implies condition (4.6). For sufficiency, let us consider $s$ arbitrary in $S$, $q$ arbitrary in $Q$ and $\delta$ arbitrary in $\mathcal{D}$. Supposing condition (4.6) holds, inclusion (4.7) has to be proved. We have that,

$$s = \sum_{i=1}^{\mu} \alpha_i v_i, \quad q = \sum_{j=1}^{\nu} \beta_j w_j, \quad \delta = \sum_{k=1}^{\eta} \rho_k r_k,$$

with $\sum_{i=1}^{\mu} \alpha_i = 1$, $\sum_{j=1}^{\nu} \beta_j = 1$, $\sum_{k=1}^{\eta} \rho_k = 1$ for some $0 \leq \alpha_i \leq 1$, $i = 1, \ldots, \mu$, $0 \leq \beta_j \leq 1$, $j = 1, \ldots, \nu$ and $0 \leq \rho_k \leq 1$, $k = 1, \ldots, \eta$. We first prove that,

$$F(q) v_i + E(q) r_k \in T_S(v_i), \quad i = 1, \ldots, \mu, \ q \in Q, \ k \in \{1, \ldots, \eta\}. \quad (4.8)$$

Let $w_{lj}$ be the $l$-th entry of $w_j$. We have that,

$$F(q) v_i + E(q) r_k = (F_0 + \sum_{l=1}^{p} F_l q_l) v_i + (E_0 + \sum_{l=1}^{p} E_l q_l) r_k$$

$$= (F_0 + \sum_{l=1}^{p} F_l \sum_{j=1}^{\nu} \beta_j w_{lj}) v_i + (E_0 + \sum_{l=1}^{p} E_l \sum_{j=1}^{\nu} \beta_j w_{lj}) r_k$$

$$= \sum_{j=1}^{\nu} \beta_j [(F_0 + \sum_{l=1}^{p} F_l w_{lj}) v_i] + \sum_{j=1}^{\nu} \beta_j [(E_0 + \sum_{l=1}^{p} E_l w_{lj}) r_k]$$

$$= \sum_{j=1}^{\nu} \beta_j [F(w_j) v_i + E(w_j) r_k], \ i \in \{1, \ldots, \mu\}, \ k \in \{1, \ldots, \eta\}.$$
From (4.6) we have that,
\[ \sum_{j=1}^{\nu} \beta_j \left[ F(w_j) v_i + E(w_j) r_k \right] \in T_S(v_i), \quad i = 1, \ldots, \mu, \quad k = 1, \ldots, \eta, \]
therefore (4.8) is proved. If \( \pi_i \) is a delimiting plane of \( S \) for \( s \) (i.e. such that \( g_i^T s = \xi_i \)), we may write \( s \) as a convex combination of the vertices of \( S \) that belong to \( \pi_i \):
\[ s = \sum_{h=1}^{\mu_i} \alpha_h v_h, \]
with \( g_i^T v_h = \xi_i \) and,
\[ \sum_{h=1}^{\mu_i} \alpha_h = 1, \quad 0 \leq \alpha_h \leq 1, \quad h = 1, \ldots, \mu_i. \]
Then,
\[ g_i^T (F(q) s + E(q) \delta) = g_i^T (F(q) \sum_{h=1}^{\mu_i} \alpha_h v_h + E(q) \sum_{k=1}^{\eta} \rho_k r_k). \]
But from (4.8) and recalling the expression of the tangent cone when \( S \) is described in terms of its vertices, we have that \( g_i^T (F(q) v_h + E(q) r_k) \leq 0 \), that implies,
\[ g_i^T (F(q) \sum_{h=1}^{\mu_i} \alpha_h v_h + E(q) \sum_{k=1}^{\eta} \rho_k r_k) \leq 0. \]
By considering all the planes for \( s \), condition (4.7) follows immediately. 

The application of Theorem 4.6 requires the knowledge of all cones \( T_S(v_i), \ i = 1, \ldots, \mu \). An alternative solution is given by the following corollary in which the Euler approximating discrete-time system of (4.5) is involved.

4.1. Corollary. The set \( S \) is positive \( D \)-invariant for system (4.5), if and only if there exists \( \tau > 0 \) such that, for all \( v_i \in \text{vert}(S) \), \( \omega_j \in \text{vert}(Q) \) and \( r_k \in \text{vert}(D) \),
\[ v_i + \tau(F(w_j) v_i + E(w_j) r_k) \in S, \quad i = 1, \ldots, \mu, \quad j = 1, \ldots, \nu, \quad k = 1, \ldots, \eta. \] (4.9)

Proof: We prove that (4.9) is equivalent to (4.6). Let suppose that (4.9) holds. Then for some \( \alpha_i, 0 \leq \alpha_i \leq 1, \ i = 1, \ldots, \mu, \) with \( \sum_{i=1}^{\mu} \alpha_i = 1, \)
\[ v_i + \tau(F(w_j) v_i + E(w_j) r_k) = \sum_{h=1}^{\mu} \alpha_h v_h, \]
therefore,
\[ F(w_j) v_i + E(w_j) r_k = \sum_{h=1}^{\mu} \frac{\alpha_h}{\tau} (v_h - v_i) \in T_S(v_i). \]
Conversely, if,

\[ F(w_j)v_i + E(w_j)r_k = \sum_{h=1}^{\mu} \gamma_h^{(i,j)}(v_h - v_i) \in T_S(v_i), \quad \text{with} \quad \gamma_h^{(i,j)} \geq 0, \ h = 1, \ldots, \mu, \]

then put,

\[ \tau^{(i,j)} = \left( \sum_{h=1}^{\mu} \gamma_h^{(i,j)} \right)^{-1} \quad \text{and} \quad \alpha_h^{(i,j)} = \tau^{(i,j)} \gamma_h^{(i,j)}, \]

to obtain,

\[ v_i + \tau^{(i,j)}(F(w_j)v_i + E(w_j)r_k) = \sum_{h=1}^{\mu} \alpha_h^{(i,j)}v_h \in S. \]

By choosing \( \tau = \min_{ij} \{\tau^{(i,j)}\} \), (4.9) follows since \( S \) is convex.

Actually, Corollary 4.1 states that the set \( S \) is positively \( D \)-invariant for system (4.5), if and only if it is positively invariant for the Euler approximating system

\[ s(t + 1) = (I_n + \tau F(q(t)))s(t) + \tau E(q(t)) \delta(t), \]

for some \( \tau > 0 \).

To overcome the problem of the choice of \( \tau \), we introduce Theorem 4.7 that provides a condition equivalent to (4.9). The proof is analogous to that of (Blanchini 1990, Theorem 2.3). Let \( C_i \) be the convex cone defined by the delimiting planes of \( S \) that contain \( v_i \):

\[ C_i = \left\{ g_h^T s \leq \xi_h, \ \xi_h > 0, \ \text{for every} \ g_h^T \right\}, \]

4.7. THEOREM. The set \( S \) is positively \( D \)-invariant for system (4.5), if and only if, for all \( \tau > 0 \), \( v_i \in \text{vert}(S) \), \( \omega_j \in \text{vert}(Q) \) and \( r_k \in \text{vert}(D) \):

\[ v_i + \tau(F(w_j)v_i + E(w_j)r_k) \in C_i, \ i = 1, \ldots, \mu, \ j = 1, \ldots, \nu, \ k = 1, \ldots, \eta. \]

If the plane description of \( S \) is available, the next corollary whose proof directly follows from that of Theorem 4.6, holds.

4.2. COROLLARY. The set \( S \) is positively \( D \)-invariant for system (4.5), if and only if, for every \( \tau > 0 \) and every \( v_i \in \text{vert}(S), \ \omega_j \in \text{vert}(Q) \),

\[ (I_n + \tau F(w_j))v_i \in C_i^*, \ i = 1, \ldots, \mu, \ j = 1, \ldots, \nu, \]

(4.10)

where \( C_i^* \) is the cone obtained by shifting the planes of \( C_i \) as follows:

\[ C_i^* = \{ g_h^T s \leq \xi_h - \max_{jk} \{ \tau g_h^T E(w_j) r_k \} \}, \ \omega_j \in \text{vert}(Q), r_k \in \text{vert}(D), \] for every \( g_h^T : g_h^T v_i = \xi_h \} \].

4.1. REMARK. According to Theorem 4.5, conditions (4.10) and (4.2) provide us with a set of inequalities in the unknowns \( K \) defining the polytope \( K \) of all the state feedback matrices solving the DLCRP.
4.3 The visibility maintenance problem

Let $\Sigma_0 \equiv \{O_0; x_0, y_0\}$ be the fixed reference frame in $\mathbb{R}^2$ and $\Sigma_F \equiv \{O_F; x_F, y_F\}$ and $\Sigma_L \equiv \{O_L; x_L, y_L\}$ the reference frames attached to a follower robot $F$ and a leader robot $L$ (see Fig. 4.1). The robots are supposed to have single integrator dynamics,

$$\dot{p}_F = \sigma_F, \quad \dot{p}_L = \sigma_L, \quad \dot{\theta}_F = \omega_F, \quad \dot{\theta}_L = \omega_L,$$

(4.11)

where $p_F = (x_F, y_F)^T$, $p_L = (x_L, y_L)^T$ are the positions, $\sigma_F = (\sigma_F^1, \sigma_F^2)^T$, $\sigma_L = (\sigma_L^1, \sigma_L^2)^T$ the linear velocities and $\omega_F$, $\omega_L$ the angular velocities of robots $F$ and $L$ in the frames $\Sigma_F$ and $\Sigma_L$, respectively. We are going to derive a dynamic model describing the relative dynamics of the robots $F$ and $L$. Referring (4.11) to the frame $\Sigma_0$, we obtain,

$$\dot{p}_F^0 = R_F^0(\theta_F) \sigma_F, \quad \dot{p}_L^0 = R_L^0(\theta_L) \sigma_L,$$

where $R_F^0(\theta_F) = \begin{bmatrix} \cos \theta_F & -\sin \theta_F \\ \sin \theta_F & \cos \theta_F \end{bmatrix}$ and $R_L^0(\theta_L)$ is defined analogously. The position of robot $L$ with respect to $\Sigma_F$ is then given by $p_L^0 = R_F^0(\theta_F)(p_L^0 - p_F^0)$. Differentiating this equation, we get,

$$\dot{p}_L^0 = \dot{R}_F^0(\theta_F)(p_L^0 - p_F^0) + R_F^0(\theta_F)(\dot{R}_F^0(\theta_F) \sigma_L^0 - R_F^0(\theta_F) \sigma_F).$$

(4.12)
Since \( \dot{R}_0^F(\theta_F) = \begin{bmatrix} 0 & \omega_F \\ -\omega_F & 0 \end{bmatrix} R_0^F(\theta_F) \), we can rewrite (4.12) as,

\[
\dot{p}_L^F = \begin{bmatrix} 0 & \omega_F \\ -\omega_F & 0 \end{bmatrix} p_L^F + R_L^F(\beta) \sigma_L^L - \sigma_F^F,
\]

(4.13)

where \( \beta \triangleq \theta_L - \theta_F \) is a shorthand of \( \beta_L^F \). Collecting equation (4.13) and the relative angular dynamics of the robots together, we obtain the system,

\[
\begin{bmatrix}
\dot{p}_F^L \\
\dot{\beta}
\end{bmatrix} =
\begin{bmatrix}
-I_2 & p_L^F[2] \\
0 & -p_L^F[1]
\end{bmatrix} \begin{bmatrix}
\sigma_F^F \\
\omega_F
\end{bmatrix} +
\begin{bmatrix}
R_L^F(\beta) \\
0
\end{bmatrix} 0 +
\begin{bmatrix}
0 \\
\sigma_L^L
\end{bmatrix},
\]

(4.14)

where \( p_L^F = (p_L^F[1], p_L^F[2])^T \). For the sake of simplicity, we will suppose that robots \( F \) and \( L \) have,

\[
\sigma_F^F = (1 + v_F, 0)^T, \quad \sigma_L^L = (1 + v_L, 0)^T,
\]

(4.15)

where \(|v_F(t)| < 1, |v_L(t)| < 1\), for all \( t \geq 0 \). Substituting (4.15) in (4.14), we finally obtain the following system,

\[
\begin{bmatrix}
\dot{p}_F^L[1] \\
\dot{p}_F^L[2] \\
\dot{\beta}
\end{bmatrix} =
\begin{bmatrix}
\cos \beta - 1 \\
\sin \beta \\
0
\end{bmatrix} +
\begin{bmatrix}
-1 & p_L^F[2] \\
0 & -p_L^F[1]
\end{bmatrix} \begin{bmatrix}
v_F \\
\omega_F
\end{bmatrix} +
\begin{bmatrix}
\cos \beta & 0 \\
\sin \beta & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
v_L \\
\omega_L
\end{bmatrix},
\]

(4.16)

with state vector \( s = (p_F^L[1], p_F^L[2], \beta)^T \in \mathcal{X} \subset \mathbb{R}^2 \times \mathbb{S}^1 \), input vector \( u = (v_F, \omega_F)^T \in \mathcal{U} \subset (-1, 1) \times \mathbb{R} \) and disturbance vector \( \delta = (v_L, \omega_L)^T \in \mathcal{D} \subset (-1, 1) \times \mathbb{R} \).

In the forthcoming analysis, we will suppose that robot \( F \) is equipped with a sensor (e.g., a panoramic camera, a laser range finder, etc.) with limited sensing range. We will call visibility set of robot \( F \) any compact and convex polyhedral set \( \mathcal{S} \subset \mathcal{X} \) containing the origin in its interior. Note that the visibility set generalizes the notion of sensor footprint since it embeds not only position but also angle information.

We suppose that robot \( L \) moves along an arbitrary trajectory and the aim of robot \( F \) is to keep \( L \) always inside its visibility set \( \mathcal{S} \), while respecting the control bound \( \mathcal{U} \). By referring to system (4.16), we can formalize this problem as follows.

**4.3. Problem (Visibility maintenance problem (VMP)).** Let \( \mathcal{S} \) be the visibility set of robot \( F \) and let \( s(0) \in \mathcal{S} \). Find a control \( u(t) \) such that for all \( \delta(t) \in \mathcal{D} \), the conditions \( s(t) \in \mathcal{S} \) and \( u(t) \in \mathcal{U} \) are fulfilled for \( t > 0 \).

If we rewrite system (4.16) in the linear parametric form (4.5), then the VMP simply reduces to the DLCRP introduced in Sect. 4.2 and suitable solvability conditions can be
3. The visibility maintenance problem

derived using conditions (4.10) and (4.2). After simple matrix manipulations in (4.16),
we obtain,
\[
\begin{bmatrix}
\Delta \dot{p}_x^F[1] \\
\Delta \dot{p}_y^F[1] \\
\Delta \dot{p}_z^F[2] \\
\beta
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \frac{\cos \beta - 1}{\beta} & \Delta \dot{p}_x^F[1] \\
0 & 0 & \frac{\sin \beta}{\beta} & \Delta \dot{p}_y^F[1] \\
0 & 0 & 0 & \Delta \dot{p}_z^F[2] \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta p_x^F[1] \\
\Delta p_y^F[1] \\
\Delta p_z^F[2] \\
\beta
\end{bmatrix} +
\begin{bmatrix}
-1 & p_x^F[2] \\
0 & -\Delta p_x^F[1] - d \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
v_x \\
v_y \\
v_z
\end{bmatrix} +
\begin{bmatrix}
\cos \beta & 0 & 0 \\
0 & \sin \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
v_L \\
\omega_L
\end{bmatrix},
\]
(4.17)

which can be written in the form (4.5) with,
\[
A(q) = \begin{bmatrix}
0 & 0 & q_2 \\
0 & 0 & 1 + q_1 \\
0 & 0 & 0
\end{bmatrix}, \quad B(q) = \begin{bmatrix}
-1 & q_4 \\
0 & -d - q_3 \\
0 & -1
\end{bmatrix}, \quad E(q) = \begin{bmatrix}
1 + q_5 & 0 \\
q_6 & 0 \\
0 & 1
\end{bmatrix},
\]
(4.18)

\(q_1 = \frac{\sin \beta}{\beta} - 1, \quad q_2 = \frac{\cos \beta - 1}{\beta}, \quad q_3 = \Delta p_x^F[1], \quad q_4 = p_x^F[2], \quad q_5 = \cos \beta - 1\) and \(q_6 = \sin \beta\). We made
the following change of variables in system (4.17),

\[(p_x^F[1], p_x^F[2], \beta)^T \rightarrow (\Delta p_x^F[1], p_x^F[2], \beta)^T,\]

where \(\Delta p_x^F[1] = p_x^F[1] - d\) and \(d\) is a strictly positive constant. There are two main
reasons for this transformation: first of all, if robot \(F\) is able to keep \(L\) always inside a visibility
set displaced of \(d\) with respect to its center (with \(d > \max\{\frac{1}{2} \| s_1 - s_2 \| : s_1, s_2 \in \text{vert}(S)\}\)),
then this automatically guarantees the collision avoidance between the robots. Second, this
choice simplifies the study of the VMP when multiple leaders are considered.

Note that \(A_0, B_0\) and \(E_0\) in (4.18) (recall the notation used in Sect. 4.2) correspond
to the constant matrices obtained by linearizing system (4.16) around the equilibrium
\(s_{eq} = (d, 0, 0)^T, u_{eq} = (0, 0)^T, \delta_{eq} = (0, 0)^T\).

4.1. ASSUMPTION. For the sake of simplicity, hereafter we will suppose that,

\[
\mathcal{U} = \{(v_F, \omega_F)^T : -V_F \leq v_F \leq V_F, -\Omega_F \leq \omega_F \leq \Omega_F\},
\]
\[
\mathcal{D} = \{(v_L, \omega_L)^T : -V_L \leq v_L \leq V_L, -\Omega_L \leq \omega_L \leq \Omega_L\},
\]
(4.19)

where \(V_F < 1, V_L < 1, \Omega_F\) and \(\Omega_L\) are strictly positive constants. We will also restrict our
attention to the following visibility set,

\[
\mathcal{S} = \{((\Delta p_x^F[1], p_x^F[2], \beta)^T : -a \leq \Delta p_x^F[1] \leq a, -a \leq p_x^F[2] \leq a, -b \leq \beta \leq b\},
\]
(4.20)

where \(a > 0\) and \(b > 0\) (see Fig. 4.2).

Constraint (4.19) is motivated by the presence of saturation bounds on the driving
motors of the robots. The set (4.20) has been chosen because is computationally sim-
ple to handle and because its cross section is a coarse approximation of a disk sensor
footprint (e.g., due to a omnidirectional camera or a 360° laser scanner).
4. Visibility maintenance via controlled invariance

**Figure 4.2:** The visibility set (4.20) and the pose of the robots L and F for \((\Delta p^E_L[1], p^E_L[2], \beta)^T = (0, 0, 0)^T, d > a\).

Since the state \((\Delta p^E_L[1], p^E_L[2], \beta)^T\) is constrained in (4.20), the polyhedron \(Q \subset \mathbb{R}^6\) of system (4.17) is defined by,

\[
q_1 \in \left[\frac{\sin b}{b} - 1, 0\right], \quad q_2 \in \left[\frac{\cos b - 1}{b}, \frac{1 - \cos b}{b}\right], \quad q_3 \in [-a, a],
\]

\[
q_4 \in [-a, a], \quad q_5 \in \left[\cos b - 1, 0\right], \quad q_6 \in [-\sin b, \sin b].
\]

We are now ready to state the main result of this section.

**4.8. Theorem (Solvability of the VMP).** Choose \(U, D\) and \(S\) as in Assumption 4.1 and let \(d > a\), \(0 < b \leq \pi/2\). The VMP for the robots F and L has a solution if the following conditions are satisfied,

\[
V_F \geq V_L \left(1 + \frac{a \sin b}{d - a}\right) + 1 - \cos b + \frac{ab}{d - a},
\]

\[
\Omega_L \leq \frac{(1 - V_L) \sin b}{d + a}, \quad \frac{V_L \sin b + b}{d - a} \leq \Omega_F.
\]

**Proof:** Let us apply Corollary 4.2 to system (4.17). By selecting \(\tau = 1\) in (4.10), we obtain,

\[
\begin{bmatrix}
1 - k_{11} + q_4 k_{21} & -k_{12} + q_4 k_{22} & q_2 - k_{13} + q_4 k_{23} \\
-(d + q_3) k_{21} & 1 - (d + q_3) k_{22} & 1 + q_1 - (d + q_3) k_{23} \\
-k_{21} & -k_{22} & 1 - k_{23}
\end{bmatrix}
\begin{bmatrix}
v_i \in C_i^*.
\end{bmatrix}
\]

Condition (4.24) must be evaluated only on the vertices \(v_1 = (a, a, b)^T, v_2 = (a, a, -b)^T, v_3 = \ldots\).
is symmetric with respect to the origin. The cones $C_1^*, \ldots, C_4^*$ are given by,

$$\begin{align*}
C_1^* &= \{ g_1^T \overline{s} \leq 1 - \frac{V_i}{a}, \ g_3^T \overline{s} \leq 1 - \frac{V_i \sin b}{a}, \ g_5^T \overline{s} \leq 1 + \frac{\Omega}{b} \}, \\
C_2^* &= \{ g_1^T \overline{s} \leq 1 - \frac{V_i}{a}, \ g_3^T \overline{s} \leq 1 - \frac{V_i \sin b}{a}, \ g_6^T \overline{s} \leq 1 - \frac{\Omega}{b} \}, \\
C_3^* &= \{ g_1^T \overline{s} \leq 1 - \frac{V_i}{a}, \ g_4^T \overline{s} \leq 1 - \frac{V_i \sin b}{a}, \ g_5^T \overline{s} \leq 1 + \frac{\Omega}{b} \}, \\
C_4^* &= \{ g_1^T \overline{s} \leq 1 - \frac{V_i}{a}, \ g_4^T \overline{s} \leq 1 - \frac{V_i \sin b}{a}, \ g_6^T \overline{s} \leq 1 - \frac{\Omega}{b} \}.
\end{align*}$$

Because of the special structure of $B(q)$ in (4.18), we can select a simplified state feedback matrix $K = \begin{bmatrix} k_{11} & 0 & k_{22} & k_{23} \\ 0 & k_{22} & k_{23} & 1 \end{bmatrix}$, that allows us to the decouple the control inputs $v_v$ and $\omega_F$ (and visualize the polytope $\mathcal{K} \subset \mathbb{R}^3$ of all the feasible gain matrices). Rewriting (4.24) in a simplified form, the following set of linear inequalities in the variables $k_{11}, k_{22}, k_{23}$ is obtained,

$$\begin{align*}
-k_{11} + q_4 k_{22} + \frac{b}{a} q_4 k_{23} & \leq -\frac{b}{a} q_2 - \frac{V_i}{a}, \\
-(d + q_3) k_{22} - \frac{b}{a} (d + q_3) k_{23} & \leq -\frac{b}{a} (1 + q_1) - \frac{V_i \sin b}{a}, \\
-k_{11} + q_4 k_{22} - \frac{b}{a} q_4 k_{23} & \leq \frac{b}{a} q_2 - \frac{V_i}{a}, \\
-(d + q_3) k_{22} + \frac{b}{a} (d + q_3) k_{23} & \leq \frac{b}{a} (1 + q_1) - \frac{V_i \sin b}{a}, \\
-\frac{a}{b} k_{22} - k_{23} & \leq -\frac{\Omega}{b}, \\
\frac{a}{b} k_{22} - k_{23} & \leq -\frac{\Omega}{b}, \\
-k_{11} - q_4 k_{22} + \frac{b}{a} q_4 k_{23} & \leq -\frac{b}{a} q_2 - \frac{V_i}{a}, \\
k_{11} - q_4 k_{22} - \frac{b}{a} q_4 k_{23} & \leq -\frac{b}{a} q_2 - \frac{V_i}{a}.
\end{align*}$$

The admissibility condition (4.2) leads to the additional constraints,

$$\begin{align*}
k_{11} & \leq \frac{V_i}{a}, & \quad k_{22} + \frac{b}{a} k_{23} & \leq \frac{\Omega}{a}, & \quad k_{22} - \frac{b}{a} k_{23} & \geq -\frac{\Omega}{a}, \\
k_{11} & \geq -\frac{V_i}{a}, & \quad k_{22} - \frac{b}{a} k_{23} & \leq \frac{\Omega}{a}, & \quad k_{22} + \frac{b}{a} k_{23} & \geq -\frac{\Omega}{a}.
\end{align*}$$

Applying the Fourier-Motzkin elimination method (Schrijver 1986) to the inequalities (4.25)-(4.26) with the assumption that $d > a$ (in order to fix the sign of the coefficients of $k_{22}$ and $k_{23}$ in
the second and fourth expression in (4.25)) we obtain the following conditions on the variables \(a, b, d, V_F, V_L, \Omega_F, \Omega_L\) and uncertain parameters \(q_1, \ldots, q_4\),

\[
\begin{align*}
\Omega_L & \leq \frac{b(1+q_1)-V_L \sin b}{d+q_3}, \quad \frac{b(1+q_1)+V_L \sin b}{d+q_3} \leq \Omega_F, \\
V_F & \geq V_L(1 + \frac{q_4 \sin b}{d+q_3}) + b(q_2 + \frac{q_4(1+q_1)}{d+q_3}), \quad \text{for } q_4 > 0, \\
V_F & \geq V_L(1 + \frac{q_4 \sin b}{d+q_3}) - b(q_2 + \frac{q_4(1+q_1)}{d+q_3}), \quad \text{for } q_4 > 0, \\
V_F & \geq V_L + b q_2, \quad \text{for } q_4 = 0, \\
V_F & \geq V_L(1 - \frac{q_4 \sin b}{d+q_3}) + b(q_2 + \frac{q_4(1+q_1)}{d+q_3}), \quad \text{for } q_4 < 0, \\
V_F & \geq V_L(1 - \frac{q_4 \sin b}{d+q_3}) - b(q_2 + \frac{q_4(1+q_1)}{d+q_3}), \quad \text{for } q_4 < 0.
\end{align*}
\]

An appropriate selection of the parameters \(q_1, \ldots, q_4\) on the extremes of the intervals (4.21), leads us to (4.22) and (4.23).

Note that conditions (4.22) and (4.23) are necessary and sufficient for the linear uncertain system (4.17). From (4.23), we see that \(\Omega_F \geq \Omega_L\).

Once fixed the variables \(a, b, d, V_F, V_L, \Omega_F, \Omega_L\) according to (4.22) and (4.23), the polytope \(\mathcal{K}\) of all the feasible state feedback matrices is simply given by (4.25)-(4.26). By evaluating (4.25)-(4.26) on the 64 vertices of the polyhedron \(\mathcal{Q}\), we see that \(\mathcal{K}\) is defined by a set of 392 inequalities, only a small number of which (see for example Fig. 4.3) is active.

4.2. Remark. Since the polytope \(\mathcal{K}\) contains infinite gain matrices, we may use an optimal criterion to select \(K\), such as, e.g., minimizing any matrix norm. In the simulations in Sect. 4.4, we have chosen the matrix \(K = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}\) with minimum 2-norm.

4.3.1 Extension: rejection of UBB disturbances

Consider the following system,

\[
\begin{bmatrix}
[\Delta p_L^F[1]] \\
\dot{p}_L^F[2] \\
\dot{\delta}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \frac{\cos \beta - 1}{\beta} \\
0 & \frac{\sin \beta}{\beta} & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
[\Delta p_L^F[1]] \\
\dot{p}_L^F[2] \\
\dot{\delta}
\end{bmatrix} + \begin{bmatrix}
-1 & 0 & \Delta p_L^F[1] - d \\
0 & -1 & -\Delta p_L^F[1] \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
v_F \\
h_F \\
\omega_F
\end{bmatrix} + \begin{bmatrix}
\cos \beta & -\sin \beta & 0 \\
\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
v_L \\
h_L \\
\omega_L
\end{bmatrix}.
\]

With respect to system (4.17), two new components, \(h_F\) and \(h_L\), are present in the vectors \(u\) and \(\delta\). They are unknown but bounded (UBB) disturbances acting on the robots \(F\) and \(L\) (e.g., lateral wind). Our purpose here is to solve the VMP in the presence of the disturbances \(h_F, h_L\).
4.3. The visibility maintenance problem

Collecting together all the perturbations acting on the nominal system (i.e., $v_L, \omega_L, h_F$ and $h_L$), we can rewrite (4.27) as,

$$
\begin{bmatrix}
\Delta \dot{p}_F^1[1] \\
\Delta \dot{p}_F^2[2] \\
\dot{\beta}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \frac{\cos \beta - 1}{\beta} & \Delta p_F^1[1] \\
0 & 0 & \sin \beta & -1 & p_F^2[2] \\
0 & 0 & 0 & 0 & -1 - d
\end{bmatrix}
\begin{bmatrix}
\Delta p_F^1[1] \\
\Delta p_F^2[2] \\
0 - \Delta p_L^1[1] - d
\end{bmatrix}
+ \begin{bmatrix}
\cos \beta & 0 & 0 & -\sin \beta
\end{bmatrix}
\begin{bmatrix}
v_L \\
\omega_L \\
\omega_F \\
0
\end{bmatrix}
+ \begin{bmatrix}
-1 & 0 & -d
\end{bmatrix}
\begin{bmatrix}
v_F \\
\omega_F \\
h_F \\
h_L
\end{bmatrix}.
$$

Let $\mathcal{U}$ be given in (4.19) and define,

$$
\mathcal{D} = \{(v_L, \omega_L, h_F, h_L)^T: -V_L \leq v_L \leq V_L, -\Omega_L \leq \omega_L \leq \Omega_L, -H_F \leq h_F \leq H_F, -H_L \leq h_L \leq H_L\},
$$

where $H_F, H_L$ are strictly positive constants.

4.3. Corollary (Solvability of the VMP with UBB disturbances). Choose $\mathcal{U}, \mathcal{D}$ as in (4.19), (4.28) and let $d > a, 0 < b \leq \pi/2$. The VMP for the robots $F$ and $L$ in the presence of the UBB disturbances $h_F, h_L$ has a solution if the following conditions are satisfied,

$$
V_F \geq V_L \left(1 + \frac{a \sin b}{d - a}\right) + 1 - \cos b + \frac{a(H_F + H_L + b)}{d - a} + H_L \sin b,
$$

$$
\Omega_L \leq \frac{(1 - V_L) \sin b - (H_F + H_L)}{d + a}, \quad V_L \sin b + b + \frac{(H_F + H_L)}{d - a} \leq \Omega_F.
$$
Figure 4.4: VMP on a circle (top-down view): the pose of the robots L and F for 
\((\Delta p_L^F[1], \Delta p_L^F[2], \Delta \beta)^T = (0, 0, 0)^T\) is shown.

Note that because of the additional terms \(H_F\) and \(H_L\), conditions (4.29) and (4.30) are stricter than (4.22) and (4.23) and then the polytope \(\mathcal{K}\) is smaller in this case. This is evident in Fig. 4.3, where the polytope \(\mathcal{K}\) (blue) obtained for 
\(a = 0.15\) m, \(b = \pi/3\) rad, \(d = 1.6\) m, \(V_F = 0.95\) m/s, \(V_L = 0.1\) m/s, \(\Omega_F = \pi/2\) rad/s, \(\Omega_L = \pi/20\) rad/s and \(H_F = 0.2\) m/s, \(H_L = 0.1\) m/s is compared with the polytope (green) obtained with \(H_L = H_F = 0\) m/s.

4.3.2 Extension: VMP on a circle

Let us consider the following change of variables in system (4.16):

\[
(p_L^F[1], p_L^F[2], \beta)^T \rightarrow (\Delta p_L^F[1], \Delta p_L^F[2], \Delta \beta)^T,
\]

\[
(v_L, \omega_L)^T \rightarrow (v_L, \Delta \omega_L)^T,
\]

\[
(v_F, \omega_F)^T \rightarrow (v_F, \Delta \omega_F)^T,
\]

where,

\[
\]

and \(\Delta \beta = \beta - \gamma, \Delta \omega_L = \omega_L - \rho, \Delta \omega_F = \omega_F - \rho\). Parameters \(0 < \gamma < \pi/2\) and \(\rho > 0\) define the pose of robot L with respect to the frame of robot F (see Fig. 4.4). Following the same procedure detailed above, we obtain,

\[
\begin{bmatrix}
\Delta p_L^F[1] \\
\Delta p_L^F[2] \\
\Delta \beta
\end{bmatrix}
= \begin{bmatrix}
0 & \rho & q_2 - \sin \gamma \\
-\rho & 0 & q_1 + \cos \gamma \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta p_L^F[1] \\
\Delta p_L^F[2] \\
\Delta \beta
\end{bmatrix}
+ \begin{bmatrix}
-1 & q_1 + \frac{1 - \cos \gamma}{\rho} \\
0 & -q_3 - \frac{\sin \gamma}{\rho} \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
v_F \\
v_L \\
\Delta \omega_F
\end{bmatrix}
+ \begin{bmatrix}
q_5 + \cos \gamma \\
q_6 + \sin \gamma \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
v_L \\
\Delta \omega_L
\end{bmatrix}.
\]
4.3. The visibility maintenance problem

where

\[ q_1 = \frac{\sin(\Delta \beta + \gamma) - \sin \gamma}{\Delta \beta} - \cos \gamma, \quad q_4 = \Delta p^F_1[2], \]
\[ q_2 = \frac{\cos(\Delta \beta + \gamma) - \cos \gamma}{\Delta \beta} + \sin \gamma, \quad q_5 = \cos(\Delta \beta + \gamma) - \cos \gamma, \]
\[ q_3 = \Delta p^F_1[1], \quad q_6 = \sin(\Delta \beta + \gamma) - \sin \gamma. \]

Since \((\Delta p^F_1[1], \Delta p^F_2[2], \Delta \beta)^T\) is supposed to be constrained in \([-a, a]^2 \times [-b, b]\), the polyhedron \(Q\) is defined as follows:

\[ q_1 \in \left[ \frac{\sin(b+\gamma)-\sin \gamma}{b} - \cos \gamma, \frac{\sin(b-\gamma)+\sin \gamma}{b} - \cos \gamma \right], \quad q_4 \in [-a, a], \]
\[ q_2 \in \left[ \frac{\cos(b+\gamma)-\cos \gamma}{b} + \sin \gamma, \frac{-\cos(b-\gamma)+\cos \gamma}{b} + \sin \gamma \right], \quad q_5 \in \left[ \cos(b+\gamma) - \cos \gamma, \cos(b-\gamma) - \cos \gamma \right], \]
\[ q_3 \in [-a, a], \quad q_6 \in [-\sin(b-\gamma) - \sin \gamma, \sin(b+\gamma) - \sin \gamma]. \]

The proof of the next theorem is analogous to that of Theorem 4.8 and it is omitted.

4.9. THEOREM (Solvability of the VMP on a circle). Let

\[(v_F, \Delta \omega_F)^T \in [-V_F, V_F] \times [-\Omega_F, \Omega_F], \]
\[(v_L, \Delta \omega_L)^T \in [-V_L, V_L] \times [-\Omega_L, \Omega_L], \]
\[(\Delta p^F_1[1], \Delta p^F_2[2], \Delta \beta)^T \in [-a, a]^2 \times [-b, b], \]

and let \(1 - \cos \gamma > \rho a, 0 \leq b \pm \gamma \leq \pi/2\). The VMP on a circle has a solution if the following conditions are satisfied,

\[ V_F \geq V_L \left( \cos(b-\gamma) + \frac{\sin(b+\gamma)(1-\cos \gamma - \rho a)}{\sin \gamma + \rho a} \right) + \cos \gamma + \rho a \]
\[ - \frac{1-\cos \gamma - \rho a}{\sin \gamma + \rho a} (\sin(b+\gamma) - \sin \gamma + \rho a) - \cos(b+\gamma), \]
\[ \Omega_L \leq \rho \left( \frac{(1-V_L) \sin(b+\gamma)}{\sin \gamma + \rho a} - 1 \right), \quad \Omega_F \geq \rho \left( \frac{V_L \sin(b+\gamma) + \sin(b-\gamma) + \sin \gamma + \rho a}{\sin \gamma - \rho a} \right). \]

4.3.3 Extension: Chain of \(n\) robots

In this section, we extend conditions (4.22)-(4.23) to a chain of \(n \geq 2\) robots (see Fig. 4.5). The object of vehicle \(j+1\) ("follower") is to maintain the vehicle ahead (robot \(j\), "leader"), in its visibility set. Let \(a_k, b_k, d_k, k = 2, \ldots, n\) be the positive parameters defining the visibility set \(S_k\) of the \(k\)-th robot and \(0 < V_k < 1, \Omega_k > 0, k = 1, \ldots, n\) the bounds on its
linear and angular velocities. By propagating conditions (4.22) and (4.23), we come up with the following set of inequalities,

\[ V_{k+1} \geq V_k \left( 1 + \frac{a_{k+1} \sin b_{k+1}}{d_{k+1} - a_{k+1}} \right) + 1 - \cos b_{k+1} + \frac{a_{k+1} b_{k+1}}{d_{k+1} - a_{k+1}}, \quad k = 1, \ldots, n - 1, \quad (4.31a) \]

\[ \Omega_1 \leq \frac{(1 - V_1) \sin b_2}{d_2 + a_2}, \quad \Omega_n \geq \frac{V_{n-1} \sin b_n + b_n}{d_n - a_n}, \quad (4.31b) \]

\[ \frac{V_{k-1} \sin b_k + b_k}{d_k - a_k} \leq \Omega_k \leq \frac{(1 - V_k) \sin b_{k+1}}{d_{k+1} + a_{k+1}}, \quad k = 2, \ldots, n - 1, \quad (4.31c) \]

that being linear in \( V_k \) and \( \Omega_k \), can be rewritten in a more compact form as follows:

\[
\begin{bmatrix}
1 + \frac{a_2 \sin b_2}{d_2 - a_2} & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 + \frac{a_n \sin b_n}{d_n - a_n} & -1 & 0 & 0 & \cdots & 0 \\
\frac{\sin b_2}{d_2 + a_2} & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
\frac{\sin b_2}{d_2 + a_2} & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{\sin b_n}{d_n + a_n} & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & \frac{\sin b_n}{d_n - a_n} & 0 & 0 & 0 & \cdots & 0 & -1
\end{bmatrix}
\begin{bmatrix}
V_1 \\
\vdots \\
V_n \\
\Omega_1 \\
\Omega_2 \\
\vdots \\
\Omega_{n-1} \\
\Omega_n
\end{bmatrix}
\leq
\begin{bmatrix}
-1 + \cos b_2 - \frac{a_2 b_2}{d_2 - a_2} \\
\vdots \\
-1 + \cos b_n - \frac{a_n b_n}{d_n - a_n}
\end{bmatrix}
\]

(4.32)

It is easy to verify that if \( a_k = a, b_k = b, d_k = d, k = 2, \ldots, n \) and \( V_k = V, k = 1, \ldots, n \), condition (4.31c) is never satisfied. To gain more insight into condition (4.31a), let us
study how $V_1$ is related to $V_n$. The following condition is obtained,

$$V_1 \leq \frac{1}{\prod_{i=1}^{n} \left(1 + \frac{a_i \sin b_i}{d_i - a_i}\right)} V_n - \sum_{i=2}^{n} \left[\frac{1 - \cos b_i + \frac{a_i \sin b_i}{d_i - a_i}}{\prod_{k=2}^{i} \left(1 + \frac{a_k \sin b_k}{d_k - a_k}\right)}\right],$$

from which we deduce that the chain can contain only a finite number of robots and that the chain can not be closed, i.e., robot 1 can not chase robot $n$, in a cyclic pursuit fashion (Marshall et al. 2004).

### 4.4 Simulation results

Simulation experiments have been conducted to illustrate the theory and show the soundness of the proposed approach. In the simulation results reported in this section, robot L moves with velocities,

$$v_L(t) = \frac{1}{20} \sin(t) \text{ m/s}, \quad \omega_L(t) = \frac{\pi}{20} \cos(t/10) \text{ rad/s}.$$

We set $V_L = 0.1 \text{ m/s}$, $\Omega_L = \pi/15 \text{ rad/s}$, $V_F = 0.9 \text{ m/s}$, $\Omega_F = \pi/3 \text{ rad/s}$, $a = 0.4 \text{ m}$, $b = \pi/4 \text{ rad}$ and $d = 2 \text{ m}$, according to the conditions of Theorem 4.8. We chose the gain matrix in $K$ with minimum 2-norm:

$$K = \begin{bmatrix} 1.5173 & 0 & 0 \\ 0 & 0.3707 & 0.4925 \end{bmatrix}.$$

Note that since $K$ is in the interior of $\mathcal{K}$, the asymptotic stability is assured. System (4.17) has been initialized with,

$$(\Delta p_L^F[1](0), p_L^F[2](0), \beta(0))^T = (0.3285, -0.1626, 0.1071)^T.$$

Fig. 4.6(a) reports the trajectory of robot L (red) and F (blue) and the visibility set $S$ (black), (in order to have a temporal reference, the robots are drawn every two seconds). Fig. 4.6(b) shows that $\Delta p_L^F[1], p_L^F[2], \beta$ (solid), keep inside the respective bounds $\pm a, \pm a, \pm b$ (dash), as expected. Finally, Fig. 4.6(c) shows that the control inputs $v_F, \omega_F$ (solid), respect the velocity bounds $\pm V_F, \pm \Omega_F$ (dash).
Figure 4.6: (a) Trajectory of the leader and follower and visibility set $S$ (top-down view); (b) $\Delta p^F_1$, $p^F_2$, $\beta$ (solid) and bounds $\pm a$, $\pm b$ (dash); (c) $v_F$, $\omega_F$ (solid) and bounds $\pm V_F$, $\pm \Omega_F$ (dash).
4.5 Conclusions and future work

In this chapter we have proposed an original solution to the visibility maintenance problem (VMP) for a leader-follower pair of Dubins-like vehicles. By interpreting the nonlinear model describing the relative dynamics of the robots as a linear system with parameter uncertainty, the VMP is reformulated as a linear constrained regulation problem with additive disturbances (DLCRP). General conditions for the positive $D$-invariance of linear uncertain systems with parametric disturbance matrix are derived and used to study the feasibility of the VMP when box bounds on the state, input and disturbance are considered. The proposed design procedure can be easily adapted to provide the control with UBB disturbances rejection capabilities. Conditions for the solution of the VMP on a circle are presented and the extension to chains of $n$ robots is also discussed.

A drawback of the procedure we have suggested is that it requires a great amount of computational work off-line, because the feedback matrix $K$ is obtained as a solution of a large set of inequalities. In addition all the vertices of the visibility set $S$ are required. From this point of view, the solution to the LCRP proposed in (Vassilaki and Bitsoris 1989) appears to be preferable to (Blanchini 1990), even though neither disturbances nor model parametric uncertainty are considered therein.

Future research lines include the extension of our results to vehicles with more involved dynamics and to general robotic networks described by directed graphs. The application of the proposed approach to the study of consensus, rendezvous and coverage problems in the presence of visibility constraints, is also a subject of on-going research.

Appendix: the Fourier-Motzkin elimination method

The Fourier-Motzkin elimination (FME), a generalization of Gauss elimination, is a computational method for solving a system

$$Ax \leq b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m,$$

of $m$ linear inequalities in $n$ variables (Schrijver 1986). The problem of solving a set of linear inequalities corresponds to determining whether the intersection polytope of $m$ halfspaces of $\mathbb{R}^n$ is nonempty and can be efficiently solved in polynomial time using the Simplex method. In fact, we can easily reformulate the problem as an linear programming problem in the variables $t$ and $x$,

$$\min_{t,x} \ t \quad \text{s.t.} \quad a_i^T x \leq b_i + t, \quad i = 1, \ldots, m.$$
If the minimizer \((x^*, t^*)^T\) satisfies \(t^* \leq 0\), then \(x^*\) satisfies the inequalities.

The key idea of the FME method is to eliminate one variable of the system \(Ax \leq b\) at each iteration and rewrite the resulting equations accordingly. Despite the number of variables decreases at each step, the number of inequalities in the remaining variables grows exponentially: in fact, at iteration \(k\) the number of inequalities to be evaluated is at most \(\lceil \frac{m}{2} \rceil^{2k}\). Because of the high spatial complexity the FME method can be applied efficiently only to problems with a small number of inequalities. In some cases, however, the method is a very powerful tool because, differently from the Simplex method, it is constructive. In fact if a solution exists, it yields a representation of the convex intersection polytope. This representation can be used to provide a symbolic solution of the set of inequalities that is useful in many applications (see the proof of Theorem 4.8).
Chapter 5

Conclusions and future research

We are at the very beginning of time for the human race. It is not unreasonable that we grapple with problems. But there are tens of thousands of years in the future. Our responsibility is to do what we can, learn what we can, improve the solutions, and pass them on.

R.P. Feynman

This final chapter contains a concise summary of the thesis’ primary contributions and ends with a discussion about future research directions.

5.1 Summary of contributions

The contributions of this thesis can be organized into two categories: those pertaining to leader-follower formation control and those pertaining to visibility maintenance.

Leader-follower formation control

Chapter 2 offers the following contributions:

1. The effect of robots’ input constraints on the set of possible trajectories of the leader and admissible positions of the followers with respect to the leader, is studied.

2. A new class of multirobot formations, called hierarchical formations and characterized by elementary leader-follower units interconnected as the nodes of a rooted tree graph, is introduced (Sect. 2.2). Recursive formulae for the maximum velocity and curvature allowed to the main leader, such that the unicycles asymptotically achieve the desired formation while respecting their input constraints, are provided (Sect. 2.4).
Chapter 3 offers the following contributions:

1. Unicycle robots equipped only with uncalibrated panoramic sensors are considered. Each camera only provides the view-angle to the other robots but not the distance, that is estimated by the extended Kalman filter.

2. The vision-based localization problem for the leader-follower formation is addressed by using a new observability condition valid for general nonlinear systems and based on the Extended Output Jacobian. Thanks to this observability condition, those trajectories of the leader that preserve system observability can be identified (Sect. 3.3).

3. An input-state feedback control law is designed to stabilize the robots to a desired formation (Sect. 3.4).

4. The critical case of distant robots is addressed by designing a feedback control via dynamic extension (Sect. 3.6).

5. A globally convergent reduced-order range estimator based on the Immersion and Invariance (I&I) technique is proposed as an alternative to the extended Kalman filter, and the stability of the closed-loop system arising from the combination of the new observer and the input-state feedback controller is proved (Sect. 3.7).

Visibility maintenance

Chapter 4 offers the following contributions:

1. The visibility maintenance problem (VMP) for a leader-follower pair of Dubins-like vehicles with input constraints, is studied.

2. The VMP is reformulated as a linear constrained regulation problem with additive disturbances (DLCRP), with the leader robot playing the role of an external disturbance. Analytical conditions for the solution of the VMP are obtained by symbolically solving with the Fourier-Motzkin elimination method, the set of linear inequalities defining the polytope of all the feasible state feedback matrices (Sect. 4.3).

3. New positive $D$-invariance conditions for general linear uncertain systems with parametric disturbance matrix are introduced (Sect. 4.2). These conditions generalize previous results due to Blanchini.
4. The proposed design procedure is computationally demanding, but general and flexible: several modified versions of the basic problem are in fact presented and discussed (Sects. 4.3.1–4.3.3).

5.2 Future research directions

The work described in this thesis generates several possible directions for future research. A few of these ideas are reported below:

- The study of decentralized versions of the control strategies proposed in Chapters 2 and 3, so that the desired formation is achieved with no (or possibly reduced) data exchange between the robots.
- The extension of the results in Chapters 2 and 3 to formations that change over time.
- The inclusion of communication/visibility constraints in the setups of Chapters 2 and 3.
- The integration of different sensor typologies, such as, e.g., laser range finders, inertial measurement units (IMUs), compassed, in the setup considered in Chapter 3.
- The study of more general visibility sets (such as, e.g., disks, circular sectors) in Chapter 4.
- The use of polar coordinates (that seem to be more appropriate than the Cartesian to depict the footprint of a range sensor with limited angular visibility), to describe the relative dynamics of the leader and follower in Chapter 4.
- The experimental validation of the control strategies in Chapters 2 and 4.
- The extension of the results of the thesis to general robotic networks.
- The extension of the results of the thesis to nonholonomic vehicles with more involved kinematics (such as, e.g., car-like robots).

To conclude this section, it is worth drawing attention to some original notions that have been recently introduced in multiagent literature and that are expected to play an important role in future research. Moving from (Eren et al. 2002, Olfati-Saber and Murray 2002), Anderson and coworkers have started to systematically apply the
5. Conclusions and future research

Figure 5.1: Rigid and nonrigid formations: The formation represented in (a) is not rigid, since it can be deformed by a smooth motion without affecting the distance between the agents connected by edges. The formations represented in (b) and (c) are rigid, since they cannot be deformed. In particular, the formation in (b) is minimally rigid, because the removal of any edge renders it nonrigid.

Rigid graph theory (Asimow and Roth 1979, Tay and Whiteley 1985) to the control of autonomous formations. In a long series of papers (see (Cao et al. 2006, Yu et al. 2007, Anderson et al. 2008) and the references therein) the authors have shown the relevance and fecundity of the notions of rigidity and persistency, that basically measure how much a formation (described by an undirected or directed graph) can be deformed by a smooth motion without affecting the distance between neighboring agents (see Fig. 5.1). Among many others, one application area where these concepts have found a natural application is sensor network localization (Aspnes et al. 2006). Lately, in (Krick et al. 2008), the notion of infinitesimal rigidity has been used to design a decentralized gradient control law for the stabilization of a group of point mass robots to a target formation.

A challenging problem that has not been addressed in the works cited above and that is the subject of on-going research, is how to apply the rigidity graph theory to the control of formations of nonholonomic vehicles.


Index

M-indistinguishability, 37
Admissible region, 65
Articulated vehicle, 28
Automated highway system, 29
Autonomous underwater vehicle, 2
Behavior based approach, 3
Bounded control, 8
Car-like robot, 28
Chi-square distribution, 62
Collective circular motion, 64
Collision avoidance, 4
Connectivity maintenance problem, 3, 63
Consensus, 3
Consistency, 61
Controlled invariance, 8, 63
Cooperative control, 1
Coverage, 3
Cyclic pursuit, 81
Deployment, 3
Disturbance decoupling problem, 10, 30
DLCRP, 8, 63, 86
Dubins-like vehicle, 63, 83, 86
Dynamic extension, 48, 86
Environmental surveillance, 2
Euler forward method, 41
Extended Kalman filter, 31, 86
Extended Output Jacobian, 7, 31, 86
Formation control, 3, 29
Fourier-Motzkin elimination, 8, 83, 86
Hierarchical formation, 7, 9, 85
Immersion and Invariance technique, 30, 31, 86
Indistinguishability, 37
Insect-like robot, 3
LCRP, 65
Leader-follower unit, 10
Lie derivative, 37
Local weak observability, 37
Localization problem, 32
Main leader, 10
Microrobot, 3
Microsatellite, 3
Model parameter uncertainty, 8, 64
Multiagent system, 1
NEES, 44
Nonholonomic vehicle, 6
Observability, 31, 36
Observability rank condition, 37
Ocean sampling, 2
Panoramic camera, 7, 31
Perspective dynamical system, 32
Pinhole camera, 35
Positive $D$-invariance, 8, 64, 86
Positive invariance, 65

Reduced-order observer, 50
Rendezvous, 3
rigid graph theory, 88
Rigidity, 88
Robotic network, 63
Rooted tree graph, 7, 10

Sensor footprint, 5
Sensor network localization, 88
Simplex method, 83
Sliding mode, 29
Structures assembling, 2
Sub-tangentiality condition, 66

Tangent cone, 66

UBB disturbances, 63, 64, 76
Unicycle, 6, 11, 33
Unmanned aerial vehicle, 2
Utilities inspection, 2

Virtual structure, 3
Visibility maintenance problem, 3, 72, 85
Visibility set, 64, 72
Vision-based localization, 31

Warehouse management, 2
Wheeled robot, 2