UNIVERSITÉ de Picardie Jules Verne

SLAM 1 – TuAT2.3

New Graph Distance Measures and Matching of Topological Maps for Robotic Exploration

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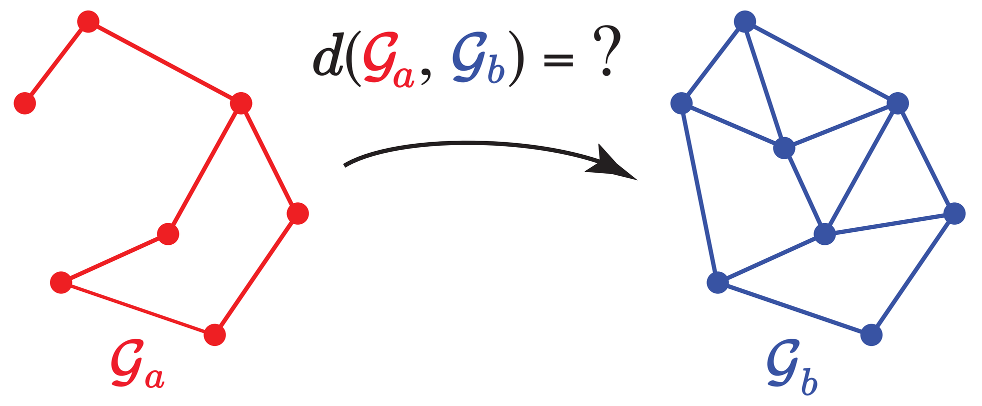
1. Motivation and original contributions

- Comparing **graph-structured maps**: crucial in *robotic exploration* and *cartography*
- Cost of the existing similarity measures, such as the *Graph Edit Distance* (GED): **Prohibitive** for large graphs ($> 10^3$ nodes)



Three new graph distances which satisfy the axioms for a **metric**:

- `LogEig`: $d_{\log}(\mathcal{G}_a, \mathcal{G}_b)$
- Bures: $d_{\mathrm{B}}(\mathcal{G}_a, \mathcal{G}_b)$
- `Rank`: $d_{\mathrm{rk}}(\mathcal{G}_a, \mathcal{G}_b)$

2. Background

Assumption: $\mathcal{G} = (V, E)$ is a *connected* undirected graph with n nodes

Definition (Shifted Laplacian $\boldsymbol{L}'$). *The shifted Laplacian is an $n \times n$ symmetric matrix defined as $\boldsymbol{L}' = \boldsymbol{L} + \boldsymbol{J}$ where $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$ is the Laplacian matrix and $\boldsymbol{J} = \frac{1}{n}\mathbb{1}\mathbb{1}^T$*

Differently from the Laplacian $\boldsymbol{L}$, the shifted Laplacian $\boldsymbol{L}'$ is positive definite

Definition (Graph Shift Operator, GSO (Mateos *et al.*, 2019)). *The GSO associated with a graph $\mathcal{G}$ is an $n \times n$ matrix $\boldsymbol{S}$, such that $[\boldsymbol{S}]_{ij} \neq 0$ if and only if $i = j$ or $\{i, j\} \in E$*

3. Graph distance measures

Definition (`LogEig` distance). *Let $\boldsymbol{L}'_a$ and $\boldsymbol{L}'_b$ be the shifted Laplacians of the graphs $\mathcal{G}_a$ and $\mathcal{G}_b$. The `LogEig` distance between $\mathcal{G}_a$ and $\mathcal{G}_b$ is*

$$d_{\log}(\mathcal{G}_a, \mathcal{G}_b) = \sqrt{\sum_{i=1}^{n} \ln^2(\lambda_i(\boldsymbol{L}'_a, \boldsymbol{L}'_b))}$$

where $\lambda_i(\boldsymbol{L}'_a, \boldsymbol{L}'_b)$, $i \in \{1, \ldots, n\}$, are the generalized eigenvalues of $\boldsymbol{L}'_a$ and $\boldsymbol{L}'_b$

Definition (Bures distance). *Let $\boldsymbol{L}'_a$ and $\boldsymbol{L}'_b$ be the shifted Laplacians of the graphs $\mathcal{G}_a$ and $\mathcal{G}_b$. The Bures distance between $\mathcal{G}_a$ and $\mathcal{G}_b$ is*

$$d_{\mathrm{B}}(\mathcal{G}_a, \mathcal{G}_b) = \sqrt{\mathrm{trace}\left[\boldsymbol{L}'_a + \boldsymbol{L}'_b - 2((\boldsymbol{L}'_a)^{1/2}\boldsymbol{L}'_b(\boldsymbol{L}'_a)^{1/2})^{1/2}\right]}$$

where $(\boldsymbol{L}'_a)^{1/2}$ is the matrix square root of $\boldsymbol{L}'_a$

Definition (`Rank` distance). *Let $\boldsymbol{S}_a$ and $\boldsymbol{S}_b$ be the GSOs of the graphs $\mathcal{G}_a$ and $\mathcal{G}_b$. The `Rank` distance between $\mathcal{G}_a$ and $\mathcal{G}_b$ is*

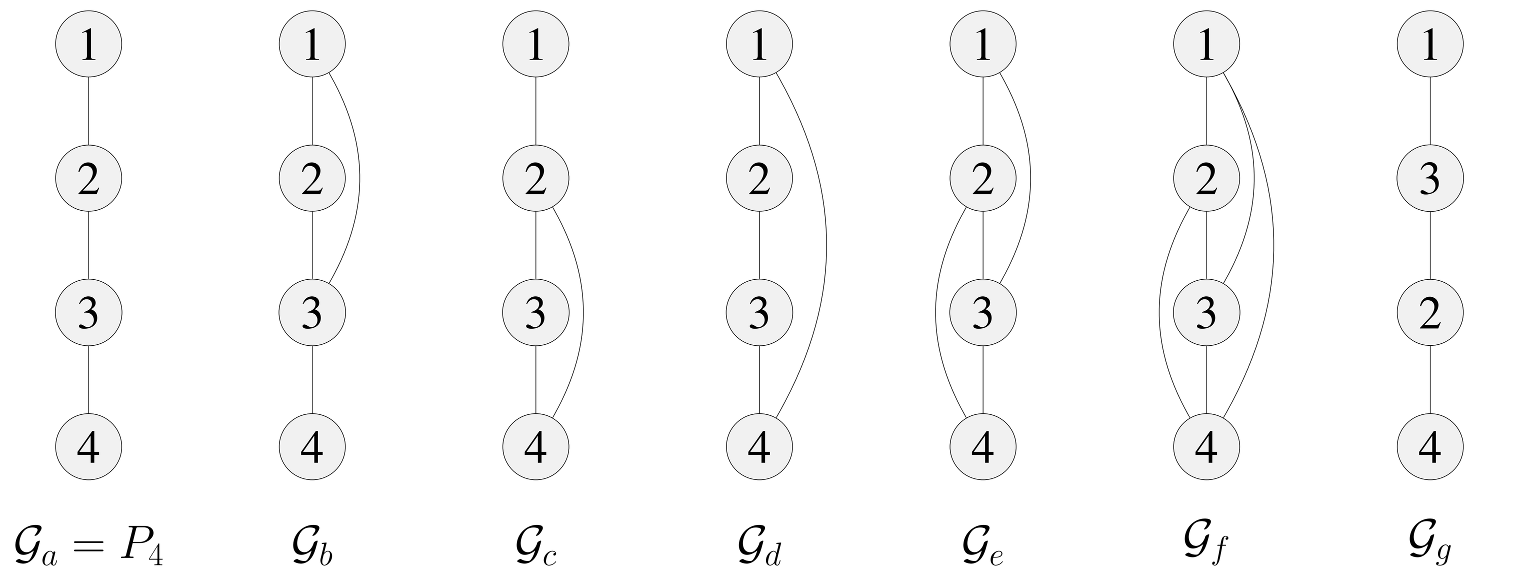
$$d_{\mathrm{rk}}(\mathcal{G}_a, \mathcal{G}_b) = \mathrm{rank}(\boldsymbol{S}_a - \boldsymbol{S}_b)$$

Measure	Graph representation	$\mathcal{G}_a$ and $\mathcal{G}_b$ not connected	If $\mathcal{G}_a \simeq \mathcal{G}_b$ $d_*(\cdot,\cdot) = 0$	Metric
GED	(V, E)	✔	✔	✔
Frobenius	$\boldsymbol{L}^s$ (symm. normal.)	✘	✘	✔
`LogEig`	$\boldsymbol{L}'$ or $\boldsymbol{L}^\star$ (reduced)	✘	✘	✔
Bures	$\boldsymbol{L}'$ or $\boldsymbol{L}^\star$ (reduced)	✘	✘	✔
`Rank`	Any GSO	✔	✘	✔

Comparison of five graph distance measures

4. Numerical experiments

4.1. Example



- As the GED, `Rank` cannot discriminate between $(\mathcal{G}_a, \mathcal{G}_b)$, $(\mathcal{G}_a, \mathcal{G}_c)$ and $(\mathcal{G}_a, \mathcal{G}_d)$. With the adjacency matrix, `Rank` is not even able to tell $(\mathcal{G}_a, \mathcal{G}_e)$ and $(\mathcal{G}_a, \mathcal{G}_f)$ apart
- Frobenius, `logEig`, Bures cannot discriminate between $(\mathcal{G}_a, \mathcal{G}_b)$ and $(\mathcal{G}_a, \mathcal{G}_c)$ *only*
- The largest Frobenius norm is obtained with the *isomorphic graphs* $\mathcal{G}_a$ and $\mathcal{G}_g$

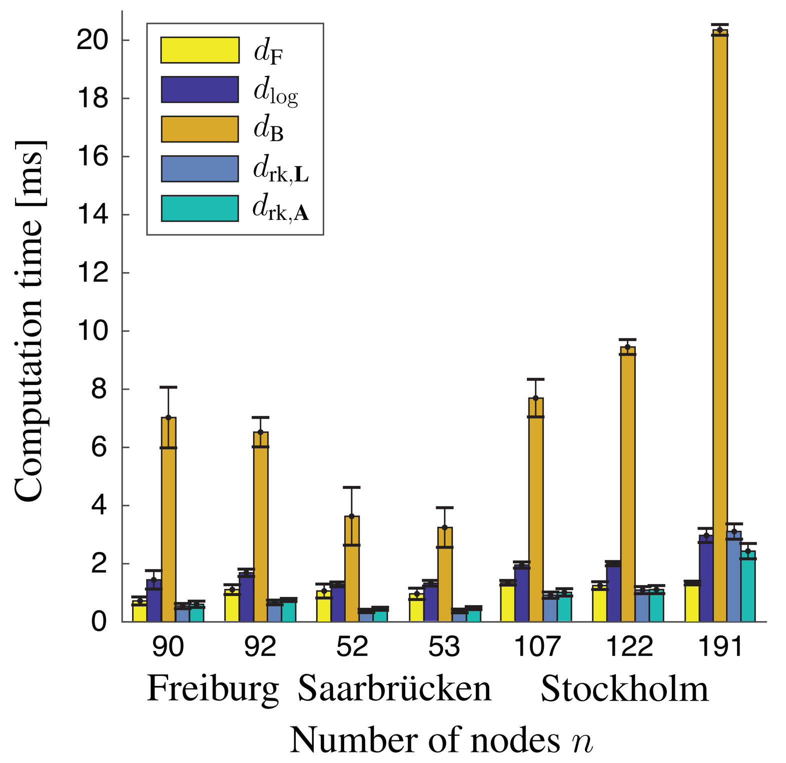
4.2. Matching of 2D topological maps

COLD-TopoMaps dataset [Zheng *et al.*, 2018]: 38 pairs of 2D topological maps

Mantel($\mathbb{D}_{\mathrm{GED}}, \cdot$)	$\mathbb{D}_{\mathrm{F}}$	$\mathbb{D}_{\log}$	$\mathbb{D}_{\mathrm{B}}$	$\mathbb{D}_{\mathrm{rk},\mathbf{L}}$	$\mathbb{D}_{\mathrm{rk},\mathbf{A}}$
Freiburg, $n = 90$	0.9973	0.9827	0.9985	0.9848	0.9753
Freiburg, $n = 92$	0.9130	0.9588	0.9556	0.9395	0.9229
Saarbrücken, $n = 52$	0.9976	0.9813	0.9955	0.9970	0.9728
Saarbrücken, $n = 53$	0.8392	0.9545	0.9395	0.7285	0.7364
Stockholm, $n = 107$	0.9826	0.9366	0.9636	0.9658	0.9892
Stockholm, $n = 122$	0.0628	0.6496	0.5960	0.3785	0.3545
Stockholm, $n = 191$	0.9332	0.9284	0.9183	0.9793	0.9702
Mean value	0.8180	**0.9131**	**0.9183**	**0.8520**	**0.8475**

Pearson coefficients $r \in [-1, 1]$ from the **Mantel test** ($M = 5000$) between $\mathbb{D}_{\mathrm{GED}}$ and $\mathbb{D}_{\mathrm{F}}$, $\mathbb{D}_{\log}$, $\mathbb{D}_{\mathrm{B}}$, $\mathbb{D}_{\mathrm{rk},\mathbf{L}}$, $\mathbb{D}_{\mathrm{rk},\mathbf{A}}$ (distance matrices of different sizes):

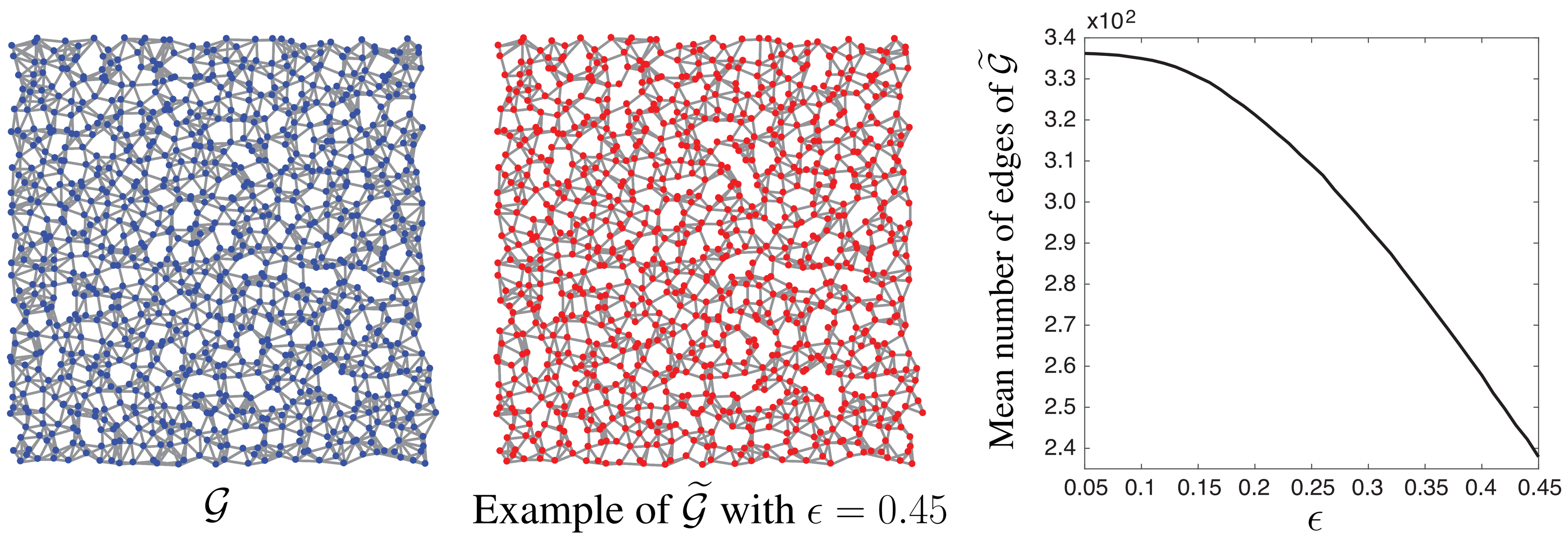
- $\mathbb{D}_{\mathrm{GED}}$ (our baseline) has a *strong relationship* with $\mathbb{D}_{\mathrm{B}}$ and $\mathbb{D}_{\log}$
- The correlation between $\mathbb{D}_{\mathrm{GED}}$ and $\mathbb{D}_{\mathrm{rk},\mathbf{L}}$, $\mathbb{D}_{\mathrm{rk},\mathbf{A}}$ is *strong* as well

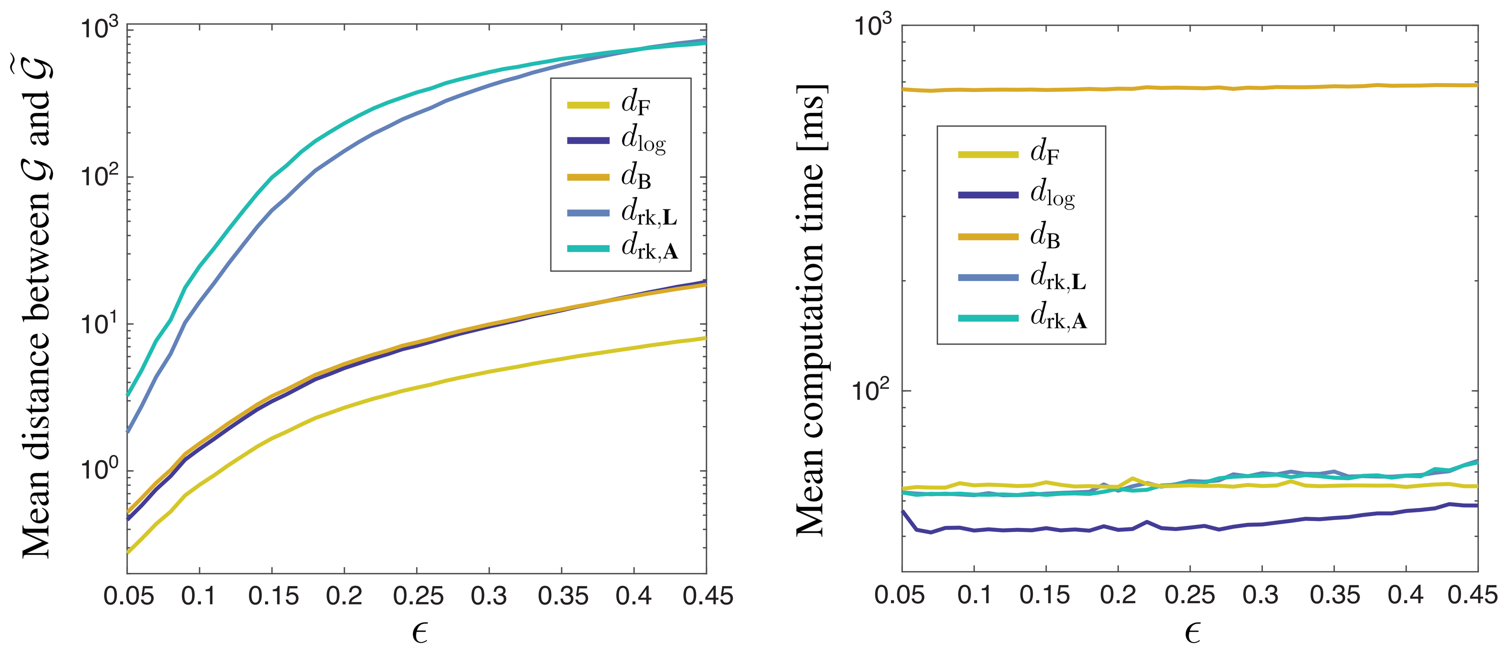


Computation time of the five metrics:

- `LogEig` and `Rank` distance: **small** (< 4 ms)
- Unlike the GED, whose calculation is *very slow*, the Bures distance can still be computed in **real time** (graphs with up to 200 nodes)

4.3. Distance between a graph $\mathcal{G}$ and its spectral sparsifier $\widetilde{\mathcal{G}}$





Spectral sparsification of a random graph $\mathcal{G}$ with 1000 nodes [Spielman & Teng, 2011]. The sparsification parameter ϵ varies between 0.05 and 0.45 (step size 0.01)

Statistics over 50 trials (*GSPBox toolbox* [Perraudin *et al.*, 2014]):

- The mean number of edges of $\widetilde{\mathcal{G}}$ decreases as ϵ grows. The larger the value of ϵ, the larger the five graph distances (as expected)
- Mean computation time:
  - ▷ `LogEig`: 41-49 ms (**best performance**)
  - ▷ `Rank` and Frobenius: 52-65 ms
  - ▷ Bures: 662 - 686 ms (**10× bigger** than the other metrics)

5. Conclusion and future work

- Three new graph distance measures (`LogEig`, Bures, `Rank`)
- The new *metrics* compare favorably with the existing distances (GED, Frobenius norm), for complex graphs in the COLD-TopoMaps dataset
- `LogEig` and `Rank` ensure *real-time performance* on a standard laptop
- We studied the effect of *spectral sparsification* on the new distance measures

- **Limitation**: $\mathcal{G}_a$ and $\mathcal{G}_b$ must have the **same number of nodes** and be **connected** (`Rank` excluded). In future works, study the *approximate graph matching problem*
- For time-varying maps, consider the *sum-rank metric* [Martínez-Peñas *et al.*, 2022]
- Use the new metrics for *fast loop-closure detection* in (topological) SLAM

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