New Graph Distance Measures and Matching of Topological Maps for Robotic Exploration

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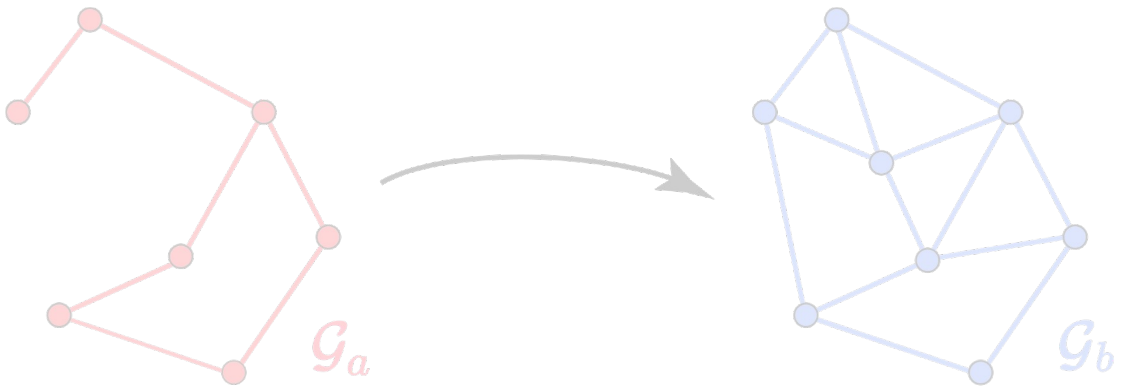
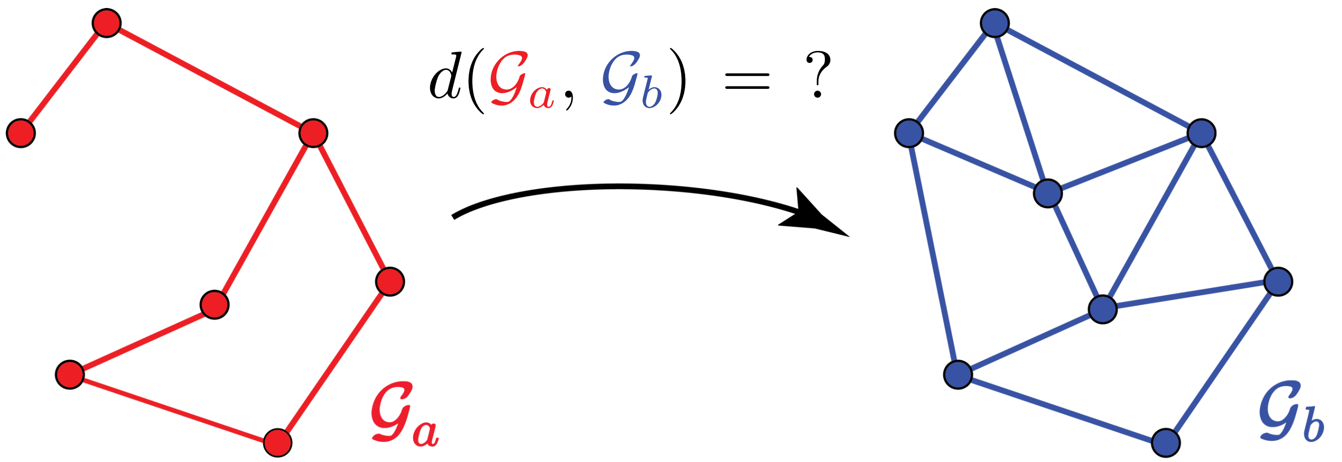


ICRA

SLAM 1 - TuAT2.3

Motivation and related work

Comparing **graph-structured maps**: crucial for *robotic exploration* and *cartography*



Problem:
Cost of the existing similarity measures, e.g. **Graph Edit Distance** (GED*):

Prohibitive for large graphs (> 1000 nodes)

*"*A survey of graph edit distance*", X. Gao, B. Xiao, D. Tao, X. Li., Pattern Analysis Applications, vol. 13, pp. 113-129, 2010

Original contributions

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Assumption:

Two *connected* **undirected** graphs $\mathcal{G}_a$ and $\mathcal{G}_b$ with n nodes

Three new graph distances:

1. **LogEig**: $d_{\log}(\mathcal{G}_a, \mathcal{G}_b) = \sqrt{\sum_{i=1}^{n} \ln^2\big(\lambda_i(\mathbf{L}'_a, \mathbf{L}'_b)\big)}$ $\mathbf{L}'_a, \mathbf{L}'_b$: Shifted Laplacians ($\mathbf{L}' = \mathbf{L} + \frac{1}{n}\mathbb{1}\mathbb{1}^T$) of $\mathcal{G}_a$ and $\mathcal{G}_b$

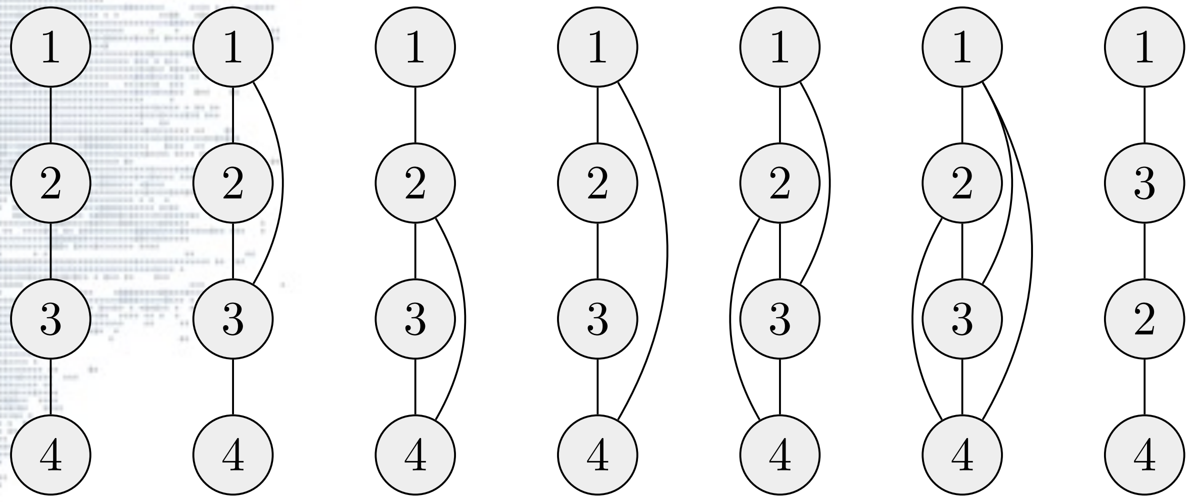
2. **Bures**: $d_{\mathrm{B}}(\mathcal{G}_a, \mathcal{G}_b) = \sqrt{\mathrm{trace}\left[\mathbf{L}'_a + \mathbf{L}'_b - 2\Big((\mathbf{L}'_a)^{1/2}\,\mathbf{L}'_b\,(\mathbf{L}'_a)^{1/2}\Big)^{1/2}\right]}$

3. **Rank**: $d_{\mathrm{rk}}(\mathcal{G}_a, \mathcal{G}_b) = \mathrm{rank}(\mathbf{S}_a - \mathbf{S}_b)$ $\mathbf{S}_a, \mathbf{S}_b$: Graph Shift Operators (GSOs) of $\mathcal{G}_a$ and $\mathcal{G}_b$

Property: The three graph distances satisfy the axioms for a **metric**

Numerical experiments

1. Example:



$\mathcal{G}_a$ $\mathcal{G}_b$ $\mathcal{G}_c$ $\mathcal{G}_d$ $\mathcal{G}_e$ $\mathcal{G}_f$ $\mathcal{G}_g$

Comparison of five metrics:

- GED (`graph_edit_distance` of *NetworkX**)
- Frobenius norm
- `LogEig`
- Bures
- `Rank` (with the Laplacian $\mathrm{rk}, \mathbf{L}$ or the adjacency matrix $\mathrm{rk}, \mathbf{A}$)

*`https://networkx.org`

$\mathbb{D}_{\mathrm{GED}}, \mathbb{D}_{\mathrm{F}}, \mathbb{D}_{\log}, \mathbb{D}_{\mathrm{B}}, \mathbb{D}_{\mathrm{rk},\mathbf{L}}, \mathbb{D}_{\mathrm{rk},\mathbf{A}}$:

7×7 distance matrices corresponding to $\mathcal{G}_a, \mathcal{G}_b, \ldots, \mathcal{G}_g$

- GED and `Rank` cannot discriminate between **3 (or more) pairs** of graphs
- Frobenius, `logEig` and Bures cannot discriminate between **2 pairs** of graphs *only*
- The largest Frobenius norm, $[\mathbb{D}_{\mathrm{F}}]_{1,7} = 2.000$, is obtained with the **isomorphic** graphs $\mathcal{G}_a$ and $\mathcal{G}_g$

Numerical experiments

1. Example:

- More **quantitative analysis** with the **Mantel test*** (a randomization test): computation of the *correlation* between two distance matrices
- We took the GED as our "baseline" and set M = 5000. Values of the Pearson coefficient $r \in [-1, 1]$:

- $\mathrm{Mntl}(\mathbb{D}_{\mathrm{GED}}, \mathbb{D}_{\mathrm{F}}) = -0.1794$
- $\mathrm{Mntl}(\mathbb{D}_{\mathrm{GED}}, \mathbb{D}_{\mathrm{log}}) = 0.2711$
- $\mathrm{Mntl}(\mathbb{D}_{\mathrm{GED}}, \mathbb{D}_{\mathrm{B}}) = 0.4728$
- $\mathrm{Mntl}(\mathbb{D}_{\mathrm{GED}}, \mathbb{D}_{\mathrm{rk},\mathbf{L}}) = 0.3691$
- $\mathrm{Mntl}(\mathbb{D}_{\mathrm{GED}}, \mathbb{D}_{\mathrm{rk},\mathbf{A}}) = 0.2390$

Conclusion:

- $\mathbb{D}_{\mathrm{GED}}$ has a *strong relationship* with $\mathbb{D}_{\mathrm{B}}$
- The correlation between $\mathbb{D}_{\mathrm{GED}}$ and $\mathbb{D}_{\mathrm{rk},\mathbf{L}}$, $\mathbb{D}_{\mathrm{log}}$, $\mathbb{D}_{\mathrm{rk},\mathbf{A}}$ is *significant* as well

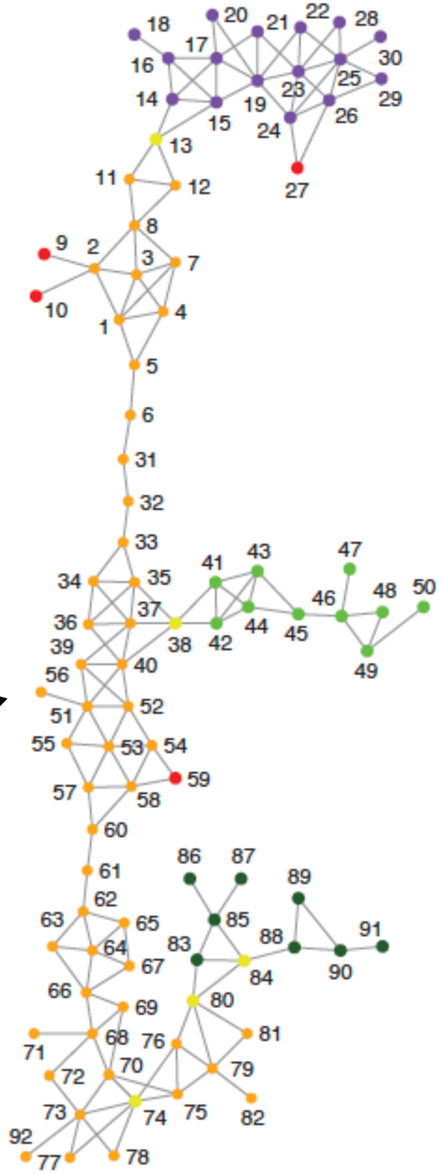
*"*Biometry: The Principles and Practices of Statistics in Biological Research*", R.R. Sokal, F.J. Rohlf, 4th edition, Freeman & Co., 2011

Numerical experiments

2. Matching of 2D topological maps

38 pairs of topological maps: **COLD-TopoMaps dataset**

- **13 pairs** with 90 and 92 nodes (Freiburg)
- **13 pairs** with 52 and 53 nodes (Saarbrücken)
- **12 pairs** with 107, 122 and 191 nodes (Stockholm)



`Seq1_sunny4`
(Freiburg: 92 nodes, 161 edges)

#	Map 1	Map 2	n
Freiburg			
1	seq1_sunny1	seq1_sunny3	90
2	seq1_sunny1	seq3_sunny1	90
3	seq1_sunny3	seq3_sunny1	90
4	seq1_cloudy2	seq1_night2	92
5	seq1_cloudy2	seq1_night3	92
6	seq1_cloudy2	seq1_sunny4	92
7	seq1_cloudy2	seq3_sunny3	92
8	seq1_night2	seq1_night3	92
9	seq1_night2	seq1_sunny4	92
10	seq1_night2	seq3_sunny3	92
11	seq1_night3	seq1_sunny4	92
12	seq1_night3	seq3_sunny3	92
13	seq1_sunny4	seq3_sunny3	92
Saarbrücken			
1	seq3_cloudy1	seq3_night1	52
2	seq3_cloudy1	seq3_night3	52
3	seq3_night1	seq3_night3	52
4	seq3_cloudy2	seq3_cloudy5	53
5	seq3_cloudy2	seq3_night2	53
6	seq3_cloudy2	seq3_sunny1	53
7	seq3_cloudy2	seq3_sunny3	53
8	seq3_cloudy5	seq3_night2	53
9	seq3_cloudy5	seq3_sunny1	53
10	seq3_cloudy5	seq3_sunny3	53
11	seq3_night2	seq3_sunny1	53
12	seq3_night2	seq3_sunny3	53
13	seq3_sunny1	seq3_sunny3	53
Stockholm			
1	floor6_base_cloudy_c	floor6_base_night_b	107
2	floor6_base_cloudy_c	floor6_base_sunny_a2	107
3	floor6_base_night_b	floor6_base_sunny_a2	107
4	floor4_cloudy_a1	floor4_night_a2	122
5	floor4_cloudy_a1	floor4_night_b	122
6	floor4_cloudy_a1	floor5_night_c	122
7	floor4_night_a2	floor4_night_b	122
8	floor4_night_a2	floor5_night_c	122
9	floor4_night_b	floor5_night_c	122
10	floor7_cloudy_a2	floor7_cloudy_b	191
11	floor7_cloudy_a2	floor7_night_a2	191
12	floor7_cloudy_b	floor7_night_a2	191

"Learning Graph-Structured Sum-Product Networks for Probabilistic Semantic Maps", K. Zheng, A. Pronobis, R.P.N. Rao, in Proc. 32nd AAAI Conf. Artif. Intell., pp. 4547-4555, 2018

Numerical experiments

2. Matching of 2D topological maps

$\mathrm{Mntl}(\mathbb{D}_{\mathrm{GED}}, \cdot)$	$\mathbb{D}_{\mathrm{F}}$	$\mathbb{D}_{\log}$	$\mathbb{D}_{\mathrm{B}}$	$\mathbb{D}_{\mathrm{rk},\mathbf{L}}$	$\mathbb{D}_{\mathrm{rk},\mathbf{A}}$
Freiburg, $n = 90$	0.9973	0.9827	0.9985	0.9848	0.9753
Freiburg, $n = 92$	0.9130	0.9588	0.9556	0.9395	0.9229
Saarbrücken, $n = 52$	0.9976	0.9813	0.9955	0.9970	0.9728
Saarbrücken, $n = 53$	0.8392	0.9545	0.9395	0.7285	0.7364
Stockholm, $n = 107$	0.9826	0.9366	0.9636	0.9658	0.9892
Stockholm, $n = 122$	0.0628	0.6496	0.5960	0.3785	0.3545
Stockholm, $n = 191$	0.9332	0.9284	0.9183	0.9793	0.9702
Mean value	0.8180	0.9131	0.9183	0.8520	0.8475

Pearson coefficients of the Mantel test with M = 5000 between $\mathbb{D}_{\mathrm{GED}}$ and $\mathbb{D}_{\mathrm{F}}, \mathbb{D}_{\log}, \mathbb{D}_{\mathrm{B}}, \mathbb{D}_{\mathrm{rk},\mathbf{L}}, \mathbb{D}_{\mathrm{rk},\mathbf{A}}$

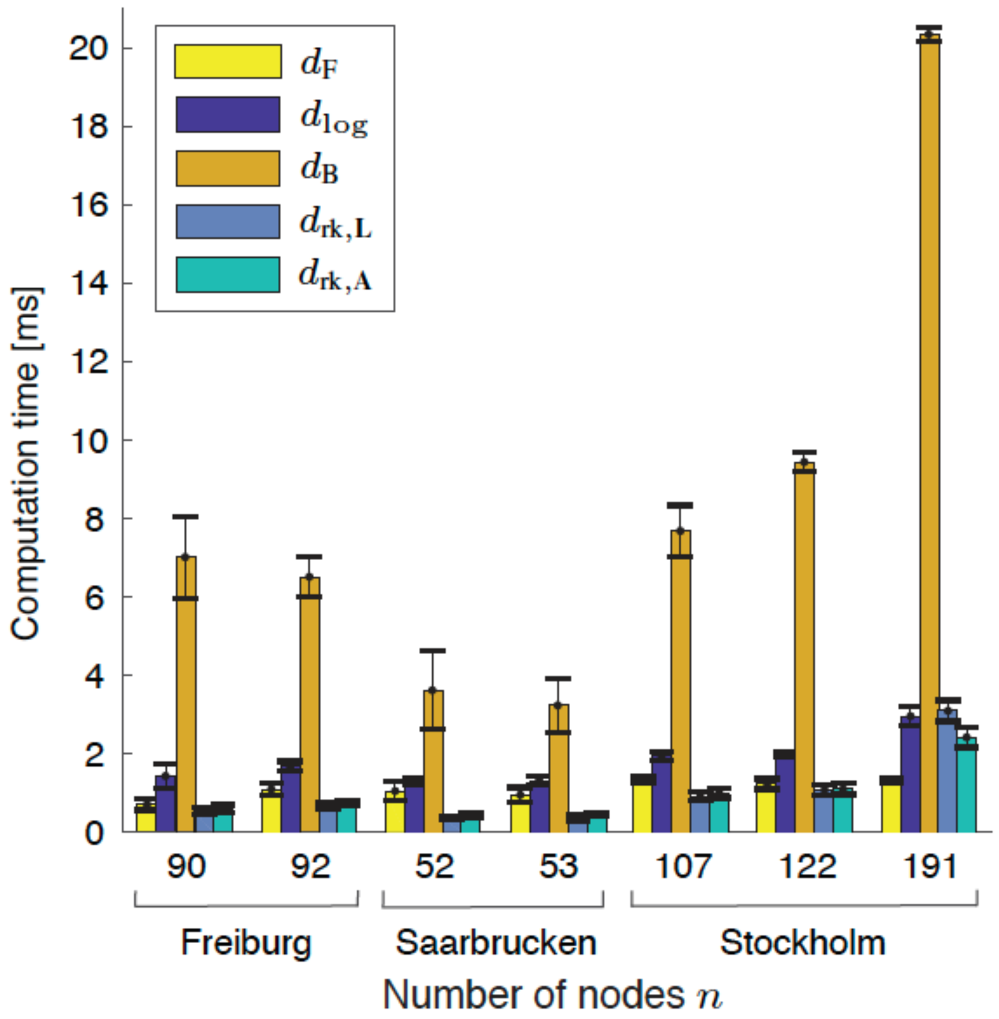
Conclusion:

- $\mathbb{D}_{\mathrm{GED}}$ has a *strong relationship* with $\mathbb{D}_{\mathrm{B}}$ and $\mathbb{D}_{\log}$
- The correlation between $\mathbb{D}_{\mathrm{GED}}$ and $\mathbb{D}_{\mathrm{rk},\mathbf{L}}$, $\mathbb{D}_{\mathrm{rk},\mathbf{A}}$ is *strong* as well, and superior to that between $\mathbb{D}_{\mathrm{GED}}$ and $\mathbb{D}_{\mathrm{F}}$

Numerical experiments

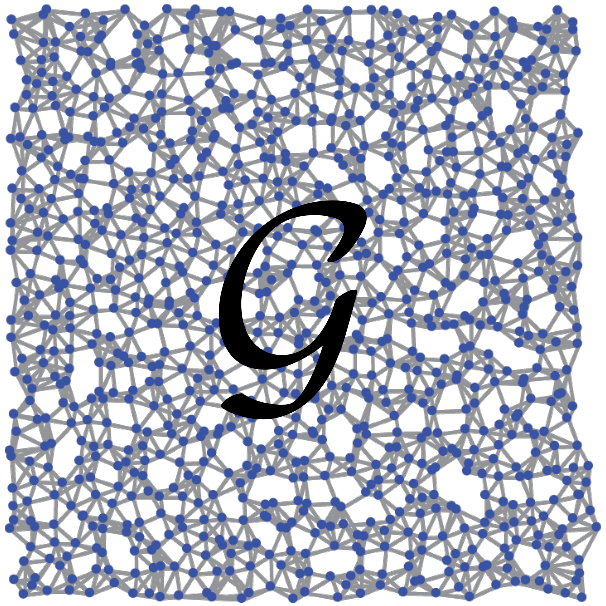
2. Matching of 2D topological maps

- **Computation time** of `LogEig` and `Rank`: small (< 4 ms)
- The calculation of the GED is *exceedingly slow* (not shown)
- The Bures distance can still be computed in real time (for maps with up to 200 nodes)

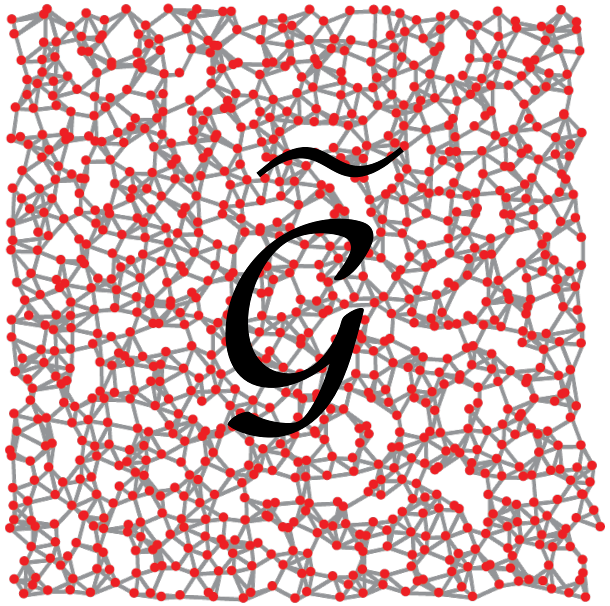


Numerical experiments

3. Distance between a graph and its spectral sparsifier



Random graph with 1000 nodes in $[0,\ 1] \times [0,\ 1]\ \mathrm{m}^2$

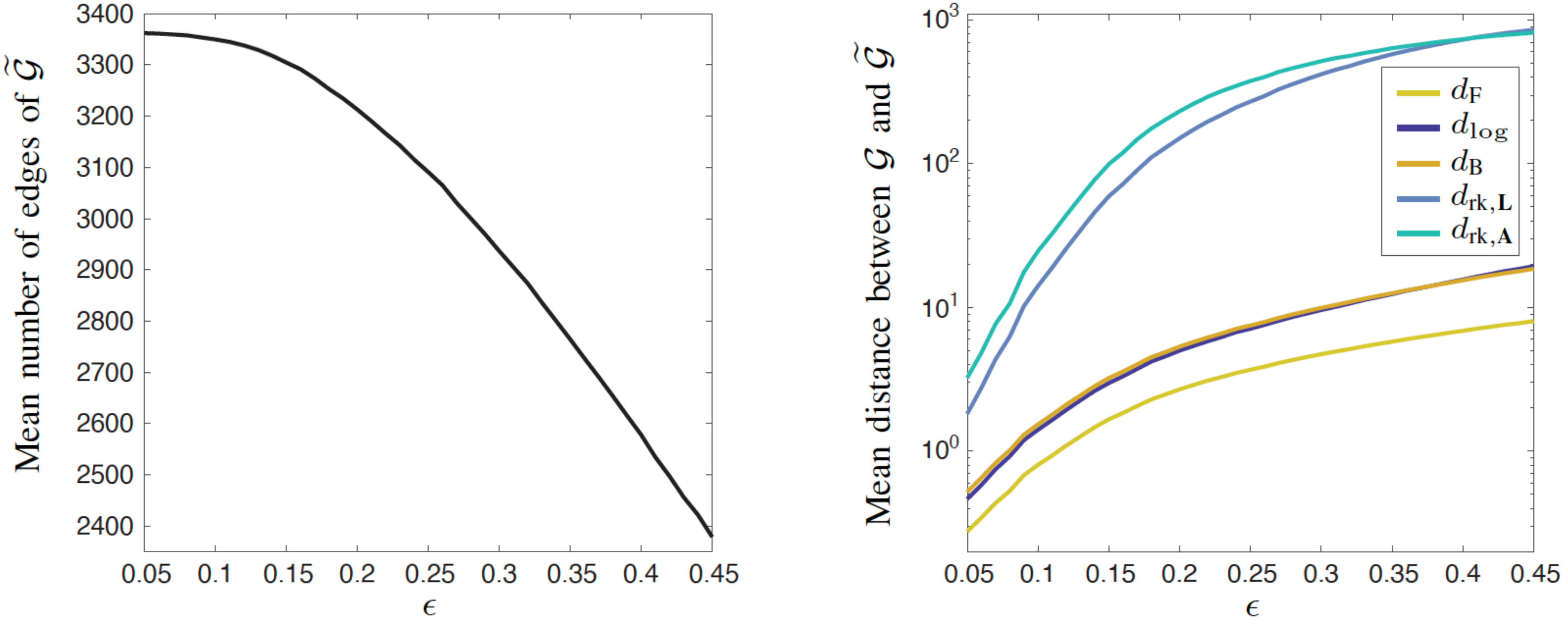


Example of spectral sparsifier with $\epsilon = 0.45$

- $\mathcal{G}$: generated with *GSPBox toolbox* [Perraudin *et al.*, 2014]
- Sparsification of $\mathcal{G}$ with Spielman & Srivastava's algorithm
- Sparsification parameter ϵ : it varies between 0.05 and 0.45 (step size 0.01)

Numerical experiments

3. Distance between a graph and its spectral sparsifier



Statistics over 50 trials:

(*Left*) The mean number of edges of the spectral sparsifier $\widetilde{\mathcal{G}}$ decreases as ϵ grows

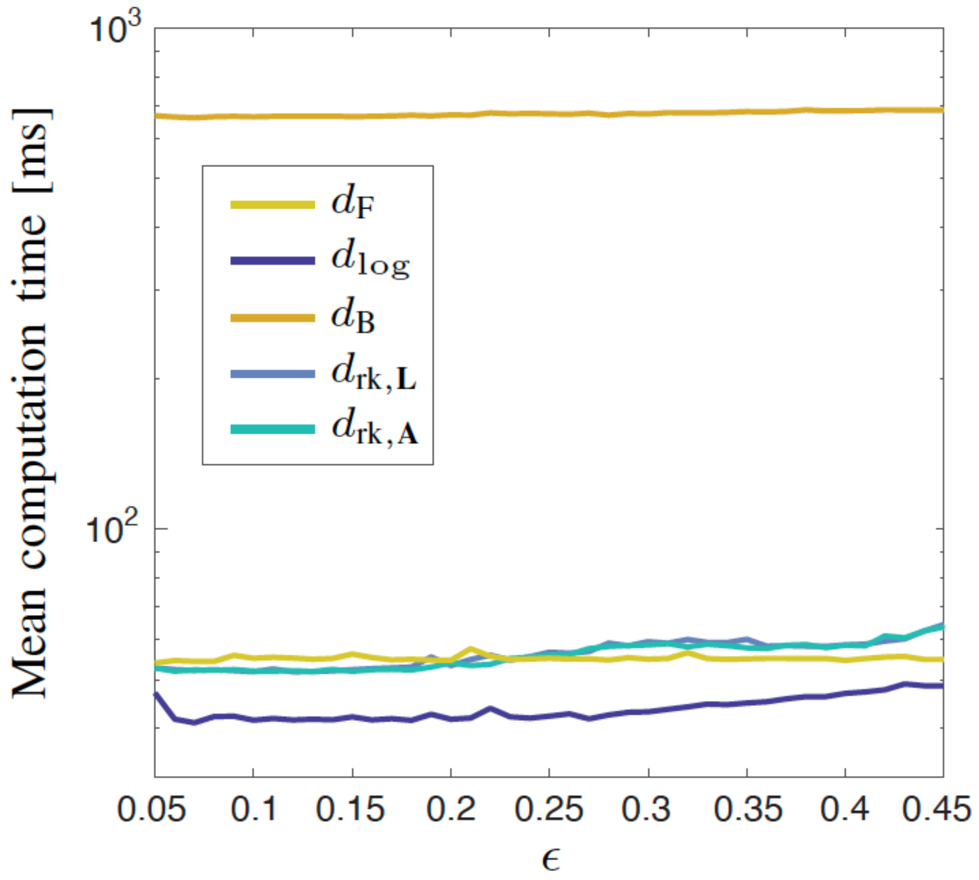
(*Right*) The mean distance between $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ increases as ϵ grows, for the *five metrics*

Numerical experiments

3. Distance between a graph and its spectral sparsifier

Mean computation time:

- `LogEig`: 41-49 ms (**best performance**)
- `Rank` and Frobenius: 52-65 ms
- Bures: 662-686 ms (**10× bigger** than the other metrics)



Conclusions and future work

- Three new graph distance measures (`LogEig`, Bures, `Rank`) which are **metrics**
- The new measures *compare favorably* with the existing distances (GED, Frobenius norm) for complex topological graphs in the COLD-TopoMaps dataset
  - `LogEig` and `Rank` ensure *real-time performance* on a standard laptop
- Study of the effect of *spectral sparsification* on the new graph distances

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- **Limitation of the new metrics**: the two graphs must have the *same number of nodes* and be *connected* (with the exception of `Rank`)
  - In future works, study the *approximate graph matching problem*
- For *time-varying maps*, consider the *sum-rank metric* [Martínez-Peñas *et al.*, 2022]
- Use the new metrics for *fast loop-closure detection* in (topological) SLAM

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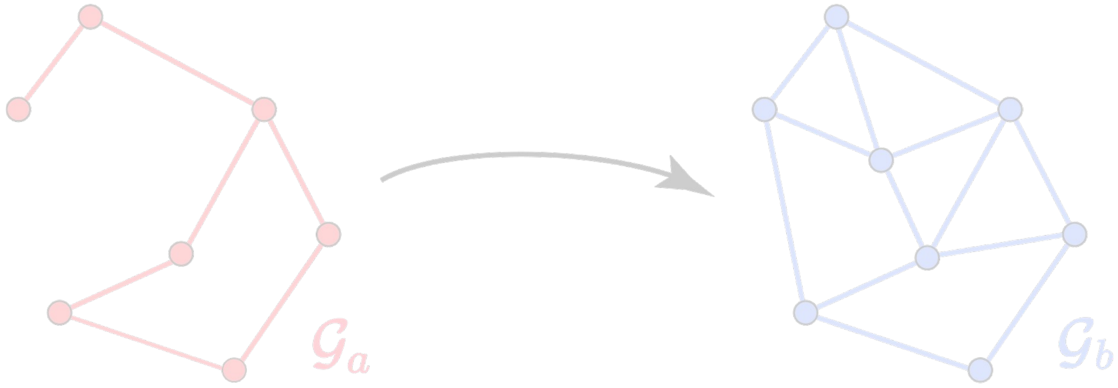

Poster TuAT2.3