How to compute Selberg-like Integrals?

Matthieu Deneufchâtel

Laboratoire d'Informatique de Paris Nord,
Université Paris 13

With J.-G. Luque, C. Carré, P. Vivo

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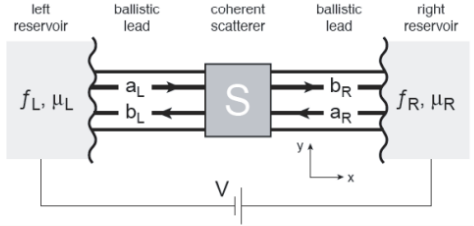
Outline

1. Physical motivation of the problem
2. Some theory about symmetric functions
   - λ-ring structure
   - Deformation of the usual scalar product
   - Jack and Macdonald polynomials
3. Simplifications and algorithm
4. Computation and asymptotics of the power sums - Selberg integral
   - Decomposition of p_μ in the Jack basis
   - Conjectures

The physical problem

- Quantum transport in chaotic cavities :



- Unitary scattering matrix S (relating the wave functions of incoming and outgoing electrons).
- Parametrisation of S :

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

(transmission t and reflection r matrices).

Random Matrix Theory Approach

- Important parameters of the system are computed with the eigenvalues x_i of $tt^\dagger$. For example :

$$\sum_{i=1}^{n} x_i \text{ (conductance);}$$

$$\sum_{i=1}^{n} x_i(1-x_i) \text{ (shot noise).}$$

- S can be fairly approximated by a random matrix belonging to a suitable ensemble (whose choice depends on the system).

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- S can be fairly approximated by a random matrix belonging to a suitable ensemble (whose choice depends on the system).
- Unitary constraint: the joint probability density for the eigenvalues x_i of the matrix is of the following form (Beenakker):

$$P(x_1,\ldots,x_N) = \frac{1}{\mathfrak{N}} \prod_{i<j}(x_i - x_j)^{2\kappa} \prod_{i=1}^{N} x_i^{\alpha-1}.$$

Integrals and Asymptotics

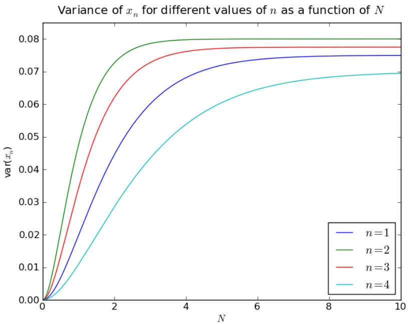
The physical quantities of interest are of the form :

$$\langle f \rangle_{a,b,\kappa}^{N} = \frac{1}{N!} \int_{[0,1]^N} f(x_1, \ldots, x_N) \prod_{i<j} (x_i - x_j)^{2\kappa} \prod_{i=1}^{N} x_i^{a-1} (1 - x_i)^{b-1} \, dx_i$$

for f a multivariate polynomial.

Two goals :

- Computation of the previous integral for several classes of functions.
- Computation of the asymptotics for $N \to \infty$ (simulating large systems).



Selberg integrals

We present an algorithm to compute

$$\langle f \rangle^{\sharp}_{a,b,\kappa,N} := \frac{\langle f \rangle^{N}_{a,b,\kappa}}{\langle 1 \rangle^{N}_{a,b,\kappa}}$$

for some multivariate polynomials f.

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Fundamental result (Kaneko) for Jack $P_\lambda^{(1/\kappa)}$ polynomials

$$\langle P_\lambda^{(\frac{1}{\kappa})} \rangle^{N}_{a,b,\kappa}$$

$$= \prod_{i<j} \frac{\Gamma\left[\lambda_i - \lambda_j + \kappa(j-i+1)\right]}{\Gamma\left[\lambda_i - \lambda_j + \kappa(j-i)\right]} \prod_{i=1}^{N} \frac{\Gamma\left[\lambda_i + a + \kappa(N-i)\right] \Gamma\left[b + \kappa(N-i)\right]}{\Gamma\left[\lambda_i + a + b + \kappa(2N-i-1)\right]}.$$

Notations

- *Sym* the algebra of symmetric functions;
- S_i the complete functions of degree i : sum over all monomials of total degree i in $x_1, \ldots, x_N$;
- p_i the power sums : $\mathrm{p}_i = \sum_{j=1}^{N} x_j^i$;
- σ_z the Cauchy kernel : $\sigma_z(\mathbb{X}) := \prod_{x \in \mathbb{X}} \frac{1}{1 - xz}$

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Property : $\sigma_z(\mathbb{X}) = \sum_i S_i(\mathbb{X}) z^i$.

λ-ring structure

Sym is endowed with a λ-ring structure :

- Sum of two alphabets :

$$\sigma_z(\mathbb{X}+\mathbb{Y}) = \sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sum_i S_i(\mathbb{X}+\mathbb{Y})z^i.$$

- Multiplication by a constant :

$$\sigma_z(u\mathbb{X}) = \sigma_z^u(\mathbb{X}),\ \forall\, u \in \mathbb{C}.$$

- Product of two alphabets :

$$\sigma_1(\mathbb{X}\mathbb{Y}) = \sum_i S_i(\mathbb{X}\mathbb{Y}) = \prod_{x\in\mathbb{X}}\prod_{y\in\mathbb{Y}} \frac{1}{1-xyt}.$$

Encoding operations that are defined on generating functions.

$\langle , \rangle$ denotes the usual scalar product on *Sym* such that

$$\langle \mathtt{p}_\mu, \mathtt{p}_\nu \rangle = \delta_{\mu,\nu}$$

with $\mathtt{p}_\mu = \prod_{i}^{\ell(\mu)} \mathtt{p}_{\mu_i}$.

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with $\mathrm{p}_\mu = \prod_i^{\ell(\mu)} \mathrm{p}_{\mu_i}$. We consider two deformations:

- a one parameter deformation $\langle , \rangle_{\frac{1}{\kappa}}$ such that

$$\langle \mathrm{p}_\mu, \mathrm{p}_\lambda \rangle_{\frac{1}{\kappa}} = z_\lambda \left(\frac{1}{\kappa}\right)^{\ell(\lambda)} \delta_{\mu,\lambda};$$

- the canonical two parameter deformation $\langle , \rangle_{q,t}$ such that

$$\langle \mathrm{p}_\lambda, \mathrm{p}_\mu \rangle_{q,t} = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \delta_{\lambda,\mu} = z_\lambda(q,t)\delta_{\lambda,\mu}$$

with $z_\lambda = \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!$.

Jack polynomials:

- defined as the unique family orthogonal *w.r.t.* $\langle , \rangle_{\frac{1}{\kappa}}$ and such that

$$P_\lambda^{\left(\frac{1}{\kappa}\right)}(\mathbb{X}) = m_\lambda(\mathbb{X}) + \sum_{\mu \leq \lambda} u_{\lambda,\mu}^\kappa m_\mu(\mathbb{X}).$$

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- obtained by applying Gram-Schmidt orthogonalization process to the Schur basis for $\langle , \rangle_{\frac{1}{\kappa}}$ beginning with 1^n.
  Algorithmic construction of a basis of symmetric functions of degree 3:

$$P_{111}^{\left(\frac{1}{\kappa}\right)} = S_{111}$$

$$P_{21}^{\left(\frac{1}{\kappa}\right)} = S_{21} - \frac{\langle S_{21}, P_{111}^{\left(\frac{1}{\kappa}\right)} \rangle_{\frac{1}{\kappa}}}{\langle P_{111}^{\left(\frac{1}{\kappa}\right)}, P_{111}^{\left(\frac{1}{\kappa}\right)} \rangle_{\frac{1}{\kappa}}} P_{111}^{\left(\frac{1}{\kappa}\right)}$$

$$P_{3}^{\left(\frac{1}{\kappa}\right)} = S_{3} - \frac{\langle S_{3}, P_{21}^{\left(\frac{1}{\kappa}\right)} \rangle_{\frac{1}{\kappa}}}{\langle P_{21}^{\left(\frac{1}{\kappa}\right)}, P_{21}^{\left(\frac{1}{\kappa}\right)} \rangle_{\frac{1}{\kappa}}} P_{21}^{\left(\frac{1}{\kappa}\right)} - \frac{\langle S_{3}, P_{111}^{\left(\frac{1}{\kappa}\right)} \rangle_{\frac{1}{\kappa}}}{\langle P_{111}^{\left(\frac{1}{\kappa}\right)}, P_{111}^{\left(\frac{1}{\kappa}\right)} \rangle_{\frac{1}{\kappa}}} P_{111}^{\left(\frac{1}{\kappa}\right)}.$$

Macdonald polynomials: defined as the unique family orthogonal *w.r.t.* $\langle , \rangle_{q,t}$ and such that

$$P_\lambda(\mathbb{X}; q, t) = m_\lambda(\mathbb{X}) + \sum_{\mu \leq \lambda} v_{\lambda,\mu}^{q,t} m_\mu(\mathbb{X}).$$

Dual basis: The dual basis of $P_\lambda(q,t)$ is the basis $Q_\lambda(q,t)$ $(Q_\lambda(q,t) = \mathfrak{C}_{\lambda,q,t} P_\lambda(q,t))$.

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Link between Jack and Macdonald polynomials: substituting q by $t^{\frac{1}{\kappa}}$ and then taking the limit $t \longrightarrow 1$, we map $P_\lambda(q,t)$ onto $P_\lambda^{(\frac{1}{\kappa})}$.

Jack-Selberg integrals

Simple operations yield the following result:

$$\langle P_\lambda^{(\frac{1}{\kappa})}\rangle^{\sharp}_{a,b,\kappa,N} = \left[\prod_{i=1}^{\ell(\lambda)} \prod_{j=i+1}^{\ell(\lambda)} \frac{\Gamma(\lambda_i - \lambda_j + \kappa(j-i+1))\Gamma(\kappa(j-i+1))}{\Gamma(\lambda_i - \lambda_j + \kappa(j-i))\Gamma(\kappa(j-i))}\right]$$
$$\left[\prod_{i=1}^{\ell(\lambda)} \prod_{j=0}^{\lambda_i - 1} \frac{\kappa(N+1-i)+j}{\kappa(\ell(\lambda)+1-i)+j}\right] \left[\prod_{i=1}^{\ell(\lambda)} \prod_{j=0}^{\lambda_i - 1} \frac{a+\kappa(N-i)+j}{a+b+\kappa(2N-i-1)+j}\right].$$

Asymptotic behavior of the Jack-Selberg integral:

$$\langle P_\lambda^{(\frac{1}{\kappa})}\rangle^{\sharp}_{a,b,\kappa,N} \underset{N\to\infty}{\sim} N^{|\lambda|}.$$

General method

Let f be a multivariate polynomial, not necessarily symmetric. The algorithm to compute $\langle f \rangle^{\sharp}_{a,b,\kappa,N}$ is the following:

1. Symmetrize f:

$$\mathfrak{S}f = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} \sigma f = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} f\left(x_{\sigma(1)}, \ldots, x_{\sigma(N)}\right)$$

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2. Expand $\mathfrak{S}f$ in the $P_\lambda^{\left(\frac{1}{\kappa}\right)}$ basis;
3. Replace $P_\lambda^{\left(\frac{1}{\kappa}\right)}$ by $\langle P_\lambda^{\left(\frac{1}{\kappa}\right)}\rangle^{\sharp}_{a,b,\kappa,N}$.

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If $\mathfrak{S}f$ admits a decomposition in the Jack basis which does not depend on the number of variables, the asymptotics can be exactly computed since the resulting integral is a rational function of N.

Specialization of Macdonald polynomials

Next step: coefficient $\alpha_{\lambda,k}$ of $P_\lambda^{(\frac{1}{\kappa})}$ in p_k?
The coefficient of $P_\lambda(q,t)$ in p_k is given by

$$(1-q^k)\frac{\prod_{\substack{(i,j)\in\lambda\\(i,j)\neq(1,1)}}(t^{i-1}-q^{j-1})}{\prod_{s\in\lambda}(1-q^{a_\lambda(s)+1}t^{l_\lambda(s)})}$$

with the *arm* and *leg* notations:

$$a_\lambda((i,j))=\lambda_i-j \text{ and } l_\lambda((i,j))=\lambda'_j-i.$$

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Found by investigating a specialization of Macdonald Q polynomials:

$$Q_\lambda(\frac{1-u}{1-t};q,t).$$

where the alphabet $\frac{1-u}{1-t}$ is considered as the difference of the alphabets

$$1+t+\cdots+t^n+\ldots \text{ and } u+tu+\cdots+t^nu.$$

The coefficient of $P_\lambda^{(\frac{1}{\kappa})}$ in p_k is equal to

$$\alpha_{\lambda,k} = k \frac{\prod\limits_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} ((j-1)-\kappa(i-1))}{\prod\limits_{s\in\lambda} (a_\lambda(s)+1+l_\lambda(s)\kappa)}.$$

This decomposition does not depend on N. Therefore, it is possible to investigate the asymptotic behavior of the power sum - Selberg integral:

$$\langle p_k \rangle^\sharp_{a,b,\kappa,N} = k \sum_{\lambda\vdash k} \frac{\prod\limits_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} ((j-1)-\kappa(i-1))}{\prod\limits_{s\in\lambda} (a_\lambda(s)+1+l_\lambda(s)\kappa)} \langle P_\lambda^{(\frac{1}{\kappa})} \rangle^\sharp_{a,b,\kappa,N}.$$

Asymptotics and conjecture

We define $\{\mathrm{p}_k\}_{a,b,\kappa} = \lim_{N\to\infty} \langle \mathrm{p}_k \rangle^{\sharp}_{a,b,\kappa,N}$.

Numerical evidences lead to the following conjectures:

Conjecture 1

If a, b, κ do not depend on N, $\{\mathrm{p}_k\}_{a,b,\kappa}$ does not depend on a, b or κ and

$$\{p_k\}_{a,b,\kappa} = \frac{1}{2^{2k}} \binom{2k}{k}.$$

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Conjecture 2

If a, b, κ are linear functions of N, $\frac{1}{N} \langle \mathrm{p}_k \rangle^{\sharp}_{a,b,\kappa,N}$ still converges and

$$\lim_{N\to\infty} \frac{1}{N} \langle p_k \rangle^{\sharp}_{\kappa(\ell-1)N,0,\kappa,N} = \frac{\ell}{(1+\ell)^{2k-1}} \sum_{i=0}^{2(k-1)} \binom{k-1}{\lceil \frac{i}{2} \rceil} \binom{k-1}{\lfloor \frac{i}{2} \rfloor} \ell^i.$$

Conclusion

- Understand the simplifications that give such a simple result;
- Understand the combinatorial structure that links these integrals to Dyck paths ($\binom{k-1}{\lceil \frac{i}{2} \rceil}\binom{k-1}{\lfloor \frac{i}{2} \rfloor}$ is the number of symmetric Dyck paths of odd semi-length k with i peaks counted by peaks);
- Generalize Newton's interpolation (we have shown that it plays a key role in the case $\kappa = 1$).