Repetition thresholds for subdivided graphs and trees

Pascal Ochem and Elise Vaslet

Non-repetitive coloring

A coloring of a graph G is non-repetitive if every non-intersecting path in G induces a square-free word.

Non-repetitive chromatic number of some graph classes:

- 3 for paths [Thue 1906]
- 3 for “large subdivisions” [Pezarski, Zmarz 2009]
- 4 for trees
- $\leq C\Delta^2$ for graphs with degree max Δ [Alon et al. 2002]
- $\leq 4^t$ for graphs with treewidth t [Kündgen, Pelsmajer 2008]
- ≥ 10 for planar graphs
- n for the complete graph on n vertices

Repetition threshold

For a graph G and an alphabet size (number of colors) k: we define the repetition threshold $RT(k,G)$.

Non-repetitive chromatic number of $G \leq k \iff RT(k,G) \leq 2$.

For a graph class $\mathcal{G}$, $RT(k,\mathcal{G}) = \sup_{G\in\mathcal{G}} RT(k,G)$.

$RT(k,\mathcal{P}) = \frac{k}{k-1}$, where $\mathcal{P}$ denotes all paths and $k \geq 5$. (Dejean's conjecture)

$RT(2,\mathcal{C}) = \frac{5}{2}$, where $\mathcal{C}$ denotes all cycles of length ≥ 18.

[Aberkane, Currie 2004]

Repetition thresholds of trees

$\mathcal{T}$ denotes the class of trees.

- $RT(2, \mathcal{T}) = \frac{7}{2}$
- $RT(3, \mathcal{T}) = \frac{5}{2}$
- $RT(k, \mathcal{T}) = \frac{3}{2}$, for $k \geq 4$.

Universal trees - Level coloring

U_n: universal family of rooted trees:

- U_0 is a single vertex.
- U_n is a root adjacent to the roots of $(k^n + 1)$ copies of U_{n-1}.

NB: $RT(k, \mathcal{T}) = RT(k, U_n)$

A rooted tree is *level colored* iff vertices with the same height have the same color.

A level coloring is specified by its coding word.

Reduction to level coloring

Claim: Every k-colored U_n contains a complete binary level colored tree of height n.

Proof: By induction on n.
U_n contains $(k^n + 1)$ subtrees U_{n-1}.
There are at most k^n ways of coloring the binary tree that U_{n-1} contains.
So two of these binary trees have the same coloring, which creates the binary tree of height n.

Conclusion: level coloring is unavoidable and is the best coloring of U_n.

Lower bounds for trees

- $RT(2,\mathcal{T}) \geq \frac{7}{2}$: Long enough binary $\frac{7}{2}$-free words contain aaa, $abab$ or $abbabba$ where $a,b \in \{0,1\}$.
- $RT(3,\mathcal{T}) \geq \frac{5}{2}$: Long enough ternary $\frac{5}{2}$-free words contain aa or aba where $a,b \in \{0,1,2\}$.
- $RT(k,\mathcal{T}) \geq \frac{3}{2}$, for $k \geq 4$: The bound $\frac{3}{2}$ comes from the star $K_{1,k}$.

Upper bounds for trees

- $k = 2$: w is the Thue-Morse word (overlap-free, 2^+-free).
- $k = 3$: w is the image by the following morphism of an infinite $\left(\frac{7}{5}^+\right)$-free word over Σ_4 starting with 0.
- $k = 4$: w is any infinite $\frac{3}{2}$-free over the 4-letter alphabet.

$$0 \mapsto 021020102120121021202101201021 2$$
$$1 \mapsto 0210201021201210201021012010212$$
$$2 \mapsto 0121012021201021012010201210212$$
$$3 \mapsto 0121012021020121012010201210212$$

Contains 02102 but avoids 20120

The suitable lemma

Let $w = w_1 w_2 \ldots w_{r-1} w_r$ be an α^+-free word and let l be an integer such that $2 \leq l \leq r$ and $w_l w_{l-1} \ldots w_2 w_1 w_2 \ldots w_{r-1} w_r$ is a repetition of exponent strictly greater than β. Then $l > \left(\frac{\beta}{\alpha} - 1\right) r + 1$.

Subdivisions

Not an actual class of graphs.
You are allowed to subdivide as much as you want in order to minimize the repetition threshold:

- $\forall$G, $\exists G_s$ a subdivision of G s.t. $RT(k, G_s) \leq \alpha$.
- $\exists$G, $\forall G_s$ a subdivision of G s.t. $RT(k, G_s) \geq \alpha$.

$\Longrightarrow$ pseudo-class $\mathcal{S}$. We set $RT(k, \mathcal{S}) = \alpha$.

Repetition thresholds of subdivisions

- $RT(2, \mathcal{S}) = \frac{7}{3}$.
- $RT(3, \mathcal{S}) = \frac{7}{4}$.
- $RT(k, \mathcal{S}) = \frac{3}{2}$, for $k \geq 4$.

Lower bounds for graph subdivisions

- $RT(2,\mathcal{S}) \geq \frac{7}{3}$: $\frac{7}{3}$-free, starting with 0: $\{00, 010, 0110\}$.
- $RT(3,\mathcal{S}) \geq \frac{7}{4}$: The bound $\frac{7}{4}$ comes from the path.
- $RT(k,\mathcal{S}) \geq \frac{3}{2}$: the bound $\frac{3}{2}$ comes from the star $K_{1,k}$.

Upper bound for subdivisions: $k = 2$

$RT(2, \mathcal{S}) \leq \frac{7}{3}$.

w is the Thue-Morse word starting at position 7,
$w = 0110010110100101100110 10\ldots$.
We use prefixes of w ending with 1011001.

Upper bound for subdivisions: $k = 3$

$RT(3, \mathcal{S}) \leq \frac{7}{4}$.

w is the image by the following morphism of an infinite $\left(\frac{7}{5}^+\right)$-free word over Σ_4.

$$0 \mapsto 02102010212012102120210120102 12$$
$$1 \mapsto 021020102120121020102101201021 2$$
$$2 \mapsto 012101202120102101201020121021 2$$
$$3 \mapsto 012101202102012101201020121021 2$$

Contains 02102 but avoids 20120.
We use factors of w starting with 02102010 and ending with 21202102.

Upper bound for subdivisions: $k \geq 4$

$RT(4, \mathcal{S}) \leq \frac{3}{2}$.

w is the image by the following morphism of an infinite $\left(\frac{7}{5}^+\right)$-free word over Σ_4.

$0 \mapsto 01321203$
$1 \mapsto 01312023$
$2 \mapsto 01231302$
$3 \mapsto 01213032$

Contains 01 but avoids 10.
We use factors of w starting with 0132 and ending with 2301.

Open problem

Is the non-repetitive chromatic number bounded for planar graphs ?