Modularity in the semilattice of $\omega$-words

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The main object of our interest is the infinite string

\[ x = x_0 x_1 x_2 \ldots x_n x_{n+1} \ldots \]

over a finite alphabet, the so called \( \omega \)-word.
Classification

- What must be classified?
- How must it be classified?
- What is the purpose of classification?
Common $\omega$–words are decimal representations of nonnegative real numbers. Here is a well-known classification:

- rational numbers,
- irrational numbers.
So we have the classification:

- ultimately periodic words,
- all other words.
Another classification comes from the theory of algorithms.

- **Reducibility by Turing machines**
  - Introduced concept of reducibility
  - Very powerful, large classes of words
- **Reducibility by automata**
  - Simpler models of computation
  - One of the simplest is the Mealy machine
  - Allows fine-grained classification of words
Why not transducers?

A transducer is a 3–sorted algebra $T = \langle Q, A, B^*; q_0, \circ, \star \rangle$ where

- $Q$, $A$, $B$ are finite, nonempty sets
  - $Q$ is the set of states
  - $A$ is the input alphabet
  - $B$ is the output alphabet
- $q_0 \in Q$ is the initial state
- $\circ : Q \times A \rightarrow Q$ is the transition function
- $\star : Q \times A \rightarrow B^*$ is the output function
Why not transducers?

A Mealy machine is a 3–sorted algebra $V = \langle Q, A, B; q_0, \circ, * \rangle$ where

$Q, A, B$ are finite, nonempty sets

- $Q$ is the set of states
- $A$ is the input alphabet
- $B$ is the output alphabet

$q_0 \in Q$ is the initial state

$\circ: Q \times A \rightarrow Q$ is the transition function

$*: Q \times A \rightarrow B$ is the output function
We say that $x$ is **Mealy reducible** to $y$ and write

$$x \rightarrow y,$$

if there exists a Mealy machine $V = \langle Q, A, B; q_0, \circ, * \rangle$ such that $y = q_0 \ast x$

Similarly, we say that $x$ is **transducible** to $y$ if there exists a transducer $T = \langle Q, A, B^*; q_0, \circ, \star \rangle$ such that $y = q_0 \star x$

**Proposition**

*If* $x$ *is transducible to* $y$ *then there exist a morphism* $\mu$ *and a Mealy machine* $\langle Q, A, B; q_0, \circ, * \rangle$ *such that* $y = \mu(q_0 \ast x)$ *where* $q_0 \in Q$
Without loss of generality, if $|A| = n$ we can choose $A = \{0, 1, \ldots, n-1\}$. Similarly, we take $B = \{0, 1, \ldots, m\}$.

Now the relation $\rightarrow$ defines a preorder on

$$\mathcal{N} = \bigcup_{n=0}^{\infty} \{0, 1, 2, \ldots, n\}$$
We say the \( \omega \)-words \( x \) and \( y \) are **Mealy equivalent** if

\[
x \rightarrow y \quad \text{and} \quad y \rightarrow x
\]

and write \( x \equiv y \)

Define the quotient set \( \tilde{\mathcal{N}} = \mathcal{N}/\equiv \) then \( \langle \tilde{\mathcal{N}}, \rightarrow \rangle \) is a poset
Structure of the Turing degrees

For $X, Y \subseteq \{0, 1, 2, \ldots \}$, $X$ is **Turing reducible** to $Y$ (i.e., $X \leq_T Y$) iff $X$ is computable using an oracle for $Y$.

The Turing degrees are the equivalence classes under $\leq_T$, ordered by $\leq_T$.

The l.u.b. of two Turing degrees is given by

$$X \oplus Y = \{2n : n \in X\} \cup \{2n + 1 : n \in Y\}.$$ 

Thus $\mathcal{D}$ (the entire collection of Turing degrees) is a semilattice.
This structure $\mathcal{D}$ has been studied extensively by recursion theorists.

**Positive results**

- For every nonzero degree $a$ there is a degree $b$ incomparable with $a$.
- There is a set of $2^{\aleph_0}$ pairwise incomparable Turing degrees.
- Every countable partially ordered set can be embedded in the Turing degrees.
Structure of the Turing degrees

Negative results

- There are minimal degrees. A degree $a$ is minimal if $a$ is nonzero and there is no degree between 0 and $a$. Thus the order relation on the degrees is not a dense order.

- There are pairs of degrees with no the greatest lower bound. Thus $\mathcal{D}$ is not a lattice.

- No infinite, strictly increasing sequence of degrees has a least upper bound.

One general conclusion that can be drawn from the research (after 50 years) is that the structure of the Turing degrees is extremely complicated.
Each downset of $\tilde{\mathcal{N}}$ defines a closed class of $\omega$-words under Mealy reducibility.

Examples of well-known classes of $\omega$-words that are closed under machine-transformations are

- automatic words (Cobham, 1972)
- morphic words (Dekking, 1994)
- periodic-like words (Muchnik et al., 2003)
- ultimately bi-ideal words (Buls, 2003)
The semilattice of $\omega$-words

**Theorem (Belovs, 2008)**

- The poset $\langle \tilde{\mathcal{N}}, \rightarrow \rangle$ is a join-semilattice.
- The width of the join-semilattice $\langle \tilde{\mathcal{N}}, \rightarrow \rangle$ is $2^{\aleph_0}$.
- The join-semilattice $\langle \tilde{\mathcal{N}}, \rightarrow \rangle$ is not atomic.
To obtain the semilattice $\langle \tilde{\mathbb{N}}, \rightarrow \rangle$ we simply take

$$[x] \lor [y] = [(x, y)]$$

The proof that there is no $\inf([x], [y])$ is not trivial and was also shown by Belovs (2008)
The semilattice of $\omega$-words

We show

**Theorem**

*The join-semilattice $\langle \tilde{\mathbb{N}}, \rightarrow \rangle$ is not modular.*

**Corollary**

*The join-semilattice $\langle \tilde{\mathbb{N}}, \rightarrow \rangle$ is not distributive.*
Distributivity

**Definition**

A join-semilattice $\langle S, \leq \rangle$ is called **distributive** when

$$\forall xab \ ( x \leq a \lor b \Rightarrow \exists a'b' \ ( a' \leq a \& b' \leq b \& x = a' \lor b' ))$$

This is analogous to

**Definition**

A lattice $\langle L, \lor, \land \rangle$ is called **distributive** when

$$\forall abc \ ( a \lor b ) \land c = ( a \land c ) \lor ( b \land c )$$
Modularity

Definition

A join-semilattice $\langle S, \leq \rangle$ is called modular when

$$\forall xab \ (a \leq x \leq a \lor b \Rightarrow \exists b' \leq b \ (x = a \lor b'))$$

This is analogous to

Definition

A lattice $\langle L, \lor, \land \rangle$ is called modular when

$$\forall xab \ (x \leq b \Rightarrow x \lor (a \land b) = (x \lor a) \land b)$$
Thank you!