An unique basis representation of finitely-generated bi-ideals

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We will concern ourselves only with right-infinite words.

- An infinite word $x$ is called **recurrent** if any finite factor of $x$ occurs in it an infinite number of times.

- A **bi-ideal sequence** from an alphabet $\Sigma$ is a sequence of words $v_0, v_1, \ldots, v_n, \ldots$, such that $v_0 \in \Sigma^+$ and $v_{i+1} \in v_i \Sigma^* v_i$.

- The infinite word we get as the limit of a bi-ideal sequence is called a **bi-ideal**.

**Theorem**

A word is recurrent if and only if it is a bi-ideal.
Finitely generated bi-ideals

There is a sequence of words \((u_n)_{n \in \mathbb{N}}\) associated to the bi-ideal sequence \((v_n)_{n \in \mathbb{N}}\), such that

\[
\begin{align*}
v_0 &= u_0 \\
v_{i+1} &= v_i u_{i+1} v_i.
\end{align*}
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We call this sequence the **base** of \((v_n)_{n \in \mathbb{N}}\).
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If a bi-ideal \(x\) is the limit of a bi-ideal sequence with base \((u_n)_{n \in \mathbb{N}}\), we say that \((u_n)_{n \in \mathbb{N}}\) is a **base** of \(x\).
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- If there exists such a basis \((u_n)_{n\in\mathbb{N}}\) for a bi-ideal \(x\), that \((u_n)_{n\in\mathbb{N}}\) is **periodic**, we say that \(x\) is **finitely-generated** and we say that \((u_0, u_1, \ldots, u_{p-1})\) is a **finite basis** of \(x\), where \(p\) is a period of \((u_n)_{n\in\mathbb{N}}\).
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- Finitely generated bi-ideals are morphic words
- The class of finitely generated bi-ideals is closed under transformations by morphisms
- The class of finitely generated bi-ideals is closed under transformations by the shift operator
- The class of finitely generated bi-ideals is not closed under transformations by 1-uniform transducers (Belovs and Buls, 2005).
Bases of bi-ideals

Lemma
There are $\aleph_0$ different bases for any bi-ideal $x$.

Example
The Thue-Morse sequence

$$t = 01101001100101101001101001\ldots$$

is a bi-ideal.
Two possible beginnings of bases for the Thue-Morse sequence are

$$B_1 = (01, 1, 0011001, \ldots)$$
$$B_2 = (0, 1101, 011001, \ldots)$$
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If \((u_0, u_1, \ldots, u_n, \ldots)\) is a base of a bi-ideal \(x\), then \((u_0 u_1, u_0 u_2, \ldots, u_0 u_n, \ldots)\) is also a base of \(x\).
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Corollary (Lorencs)

If \((u_0, u_1, \ldots, u_n)\) is a finite base of a finitely-generated bi-ideal \(x\), then

\[(u_0 u_1, u_0 u_2, \ldots, u_0 u_n, u_0 u_0)\]

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is also a finite base of \(x\).

**Corollary (Lorencs)**
There are \(\aleph_0\) different finite bases for any finitely-generated bi-ideal.
Lemma

If there exists an ultimately periodic basis for a bi-ideal $x$, then $x$ is finitely generated. T.i., there exists a finite basis for $x$.

Example

Consider the finitely generated bi-ideal

$$x = 0101001010100101001010010100101010010\ldots$$

A finite basis of this bi-ideal is, for example,

$$B_f = (01, 010),$$

while a ultimately periodic basis for $x$ would be, for example,

$$B_u = (0, 1, 10, 1, 10, 1, 10, \ldots)$$
Lemma

Every infinite periodic word $x = u^\omega$ is a finitely generated bi-ideal, a finite basis of which is $(u)$. 

Theorem (Buls and Lorencs, 2008)

A finitely generated bi-ideal $x$ with a finite basis $(u_0, u_1, \ldots, u_n)$ is periodic if and only if there exists a word $v$ and a set of exponents $k_i \in \mathbb{N}$ such that $u_i = v^{k_i}$ for all $i \in \{0, 1, \ldots, n\}$.

Example. Given the infinite periodic word $(01)^\omega$, some finite bases for it include $B_1 = (01)$, $B_2 = (01, 01, 010101)$, and $B_3 = (0101, \epsilon, 01)$.
Finitely generated bi-ideals and periodicity

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The problem

The identity problem
Given two finite bases \((u_0, u_1, \ldots, u_n)\) and \((u'_0, u'_1, \ldots, u'_m)\), corresponding to the finitely generated bi-ideals \(x\) and \(y\), respectively, — how can you tell, whether \(x = y\)?

The unique representation problem
Given a finitely generated bi-ideal \(x\), find a unique finite basis for \(x\).

Example
Do the finite bases \(B_1 = (010100101, 0101001010)\) and \(B_2 = (01010, 0101, 01010, 0101)\) generate the same bi-ideal, and if so, what would be a suitable representation for this bi-ideal?
Alternative formulation

Given two finite bases $B$ and $B'$ of finitely generated bi-ideals $x$ and $y$, provide a criterion, such that if it is true, either $x \neq y$ or $x$ and $y$ are periodic.
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Theorem (Lorencs)

Consider two finite bases $B = (u_0, u_1)$ and $B' = (u'_0, u'_1)$, for a bi-ideal $x$, such that $u'_0 > u_0$. Either $x$ is periodic or there exists $n \in \mathbb{N}$ such that $T^n(B) = B'$, where $T(u_0, u_1) = (u_0u_1, u_0u_0)$.
Previous progress

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Theorem (Lorenco)
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Theorem (Bêts)
$B = (u_0, u_1, \ldots, u_n)$ and $B' = (u'_0, u'_1, \ldots, u'_m)$ are finite bases of a bi-ideal $x$, with $\gcd(n, m) = 1$, and $\forall i, j : |u_i| = |u'_j| = \text{const.}$ Then $x$ is periodic.
The solution

- We offer three effective reductions, simplifying the base.
- Each reduction decreases the total length of the base words.
- We prove, that a completely reduced base is a unique representation of a finitely generated bi-ideal.
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- Each reduction decreases the total length of the base words.
- We prove, that a completely reduced base is a unique representation of a finitely generated bi-ideal.

This solves both the identity and the unique representation problems, since

1. it provides a unique representation for a finitely generated bi-ideal
2. to determine if two finite bases generate the same bi-ideal, we can first completely reduce them, and then compare the resulting reduced bases.
Reduction 1

Given \((u_0, u_1, \ldots u_n)\) is a finite basis for \(x\) and there exists a word \(v\) and a set of exponents \(k_i \in \mathbb{N}\) such that \(u_i = v^{k_i}\) for all \(i \in \{0, 1, \ldots, n\}\), then 

\[(v)\]

is also a basis of \(x\).

This is really just a restatement of the earlier theorem by Buls and Lorencs (2008).

Example

The bases \(B' = (01, 01, 010101)\) and \(B'' = (0101, \epsilon, 01)\) would be reduced to \(B = (01)\) by this reduction.
Base reduction

Reduction 2
Given \((u_0, u_1, \ldots u_n)\) is a finite basis for \(x\) and there exist \(k, T \in \mathbb{N}\), such that \(n = k \cdot T - 1\) and \(u_i = u_{i+T}\) for all \(i \in \{0, 1, \ldots, n - T\}\), then

\[(u_0, u_1, \ldots, u_{T-1})\]

is also a basis of \(x\).

Example
The base \(B' = (01010, 0101, 01010, 0101)\) would be reduced to \(B = (01010, 0101)\) by this reduction.
Reduction 3

Given \((u_0, u_1, \ldots, u_n)\) is a finite basis for \(x\) and there exists a set of words \(\omega_i\), such that \(u_i = \omega_n \omega_i\) for all \(i \in \{0, 1, \ldots, n\}\), then

\[(\omega_n, \omega_0, \omega_1, \ldots, \omega_{n-1})\]

is also a basis of \(x\).

This is a restatement of the Lemma by Lorencs.

Example

The base \(B'' = (01001, 01000, 010010)\) would be reduced to \(B' = (010, 01, 00)\), and this further to \(B = (0, 10, 1)\) by this reduction.
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The base \(B'' = (01001, 01000, 010010)\) would be reduced to \(B' = (0 \ 10, 0 \ 1, 0 \ 0)\), and this further to \(B = (0, 10, 1)\) by this reduction.
Our main result

Definition
A finite basis \((u_0, u_1, \ldots u_n)\) is said to be **reduced**, if neither the total length of the base words, nor the number of base words can be reduced using reductions 1, 2 or 3.
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\textit{A reduced basis is a unique representation of a finitely generated bi-ideal.}
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Theorem
A reduced basis is a unique representation of a finitely generated bi-ideal.

Corollary
A finitely generated bi-ideal \(x\) is periodic if and only if the length of its reduced basis is 1.
Example

Do the bases $B_1 = (010100101, 0101001010)$ and $B_2 = (01010, 0101, 01010, 0101)$ generate the same bi-ideal?
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- Reduce $(01010 0101, 01010 01010)$ to $(01010, 0101)$

So, $B_1$ and $B_2$ generate the same bi-ideal. Moreover, the bi-ideal is aperiodic.
Example

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- Reduce $(010100101, 0101001010)$ to $(01010, 0101)$
- Reduce $(01010, 0101)$ to $(01010, 0101)$

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- Reduce $(01010 0101, 01010 01010)$ to $(01010, 0101)$
- Reduce $(01 010, 01 01)$ to $(01, 010)$
- Reduce $(01010, 0101, 01010, 0101)$ to $(01010, 0101)$

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- Reduce $(0101, 0101)\text{ to } (0101, 0101)$
- Reduce $(01010101, 010101010, 010101010, 010101010)\text{ to } (01010, 0101)$
- Reduce $(0101, 0101)\text{ to } (0101, 0101)$

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