An application of iterative pushdown automata to contour words of balls and truncated balls in hyperbolic tessellations

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in this talk:

1. iterated pushdown automata
2. tessellations in the hyperbolic plane
3. contour words of balls
4. in 3D
1. iterated pushdown automata
we will see, successively:

1.1 pushdown automata

1.2 iterated pushdown automata
1.1 pushdown automata

it consists of two structures:

a finite automaton

finitely many states, finite alphabet
reading the input word letter after letter with possible $\epsilon$-transitions
and an additional feature:

a stack:

  list of letters, top only accessible while reading, two possible actions:

  push and pop

transitions: fixed by the read letter and by the top of the stack
1.2 iterated pushdown automata

again a finite automaton,

but with an iterated store:

again stack structure
but the elements are stacks of stacks

below, formal presentation based on a paper by G. Sénizergues and S. Fratani, 

formally define $k$-iterated store as follows:

$$0\text{-pds}(\Gamma) = \{\epsilon\}$$
$$k+1\text{-pds}(\Gamma) = (\Gamma[k\text{-pds}(\Gamma)])^*$$
$$\text{it-pds} = \bigcup_k k\text{-pds}(\Gamma)$$

unique representation of a $k+1$ store $\omega$ as:

$$\omega = A[\text{flag}].\text{rest}, \ A \in \Gamma,$$

with $\text{flag}$ k-store, and $\text{rest}$ $k+1$-store

we have: $\text{lgth}(\omega) = \text{lgth}(\text{rest}) + 1$
operations on $k$-iterated stores:

notion of top symbols:

$$\text{topsym}(\epsilon) = \epsilon,$$

$$\text{topsym}(A[\text{flag}].\text{rest}) = A.\text{topsym}(\text{flag})$$

the pop operation:

$$\text{pop}_j(\epsilon) \text{ undefined}$$

$$\text{pop}_{j+1}(A[f].r) = A[\text{pop}_j(f)].r, \quad A \in \Gamma$$
operations on $k$-iterated stores:

the **push** operation:

$$\text{push}_1(w)(\epsilon) = w, \ w \in \Gamma^*$$

$$\text{push}_j(\gamma)(\epsilon) \text{ undefined for } j > 1$$

$$\text{push}_1(w)(A[f].r) = w_1[f]..w_h[f].r,$$

$$\text{push}_{j+1}(w)(A[f].r) = A[\text{push}_j(w)(f)].r,$$

$$w = w_1..w_h \in \Gamma^*, \ w_i \in \Gamma$$

operations performed within the top symbols
$k$-iterated pushdown automata:

operate on a word as a finite automaton, using a $k$-store and the \textbf{push} and \textbf{pop} operations.

computation:

there are final states word \textbf{accepted} if and only if at least one path of computation leads to a final state.
**Theorem (G. Sénizergues, 2003)** — the languages recognized by a $k$-iterated pushdown automaton define a strictly increasing hierarchy with respect to $k$

Key example:

2-pds automata recognize:

$\{a^{f_n} \mid \{f_n\}_{n \in \mathbb{N}} : \text{Fibonacci sequence} \}$
three states: $q_0$, $q_1$ and $q_2$;  
input word in $\{a\}^*$; $\Gamma = \{Z, X_1, X_2, F\}$;  
initial state: $q_0$; initial stack: $Z[\epsilon]$;  
transition function $\delta$:  
\[
\delta(q_0, \epsilon, Z) = \{(q_0, \text{push}_2(F)), (q_0, \text{push}_1(X_2))\}
\]
\[
\delta(q_0, \epsilon, ZF) = \{(q_0, \text{push}_2(FF)), (q_0, \text{push}_1(X_2))\}
\]
\[
\delta(q_0, \epsilon, X_1F) = (q_1, \text{pop}_2)
\]
\[
\delta(q_0, \epsilon, X_2F) = (q_2, \text{pop}_2)
\]
\[
\delta(q_0, a, X_1) = (q_0, \text{pop}_1)
\]
\[
\delta(q_0, a, X_2) = (q_0, \text{pop}_1)
\]
\[
\delta(q_1, \epsilon, X_1F) = (q_0, \text{push}_1(X_1X_2))
\]
\[
\delta(q_2, \epsilon, X_2F) = (q_0, \text{push}_1(X_1))
\]
\[
\delta(q_1, \epsilon, X_1) = (q_0, \text{push}_1(X_1X_2))
\]
\[
\delta(q_2, \epsilon, X_2) = (q_0, \text{push}_1(X_1))
\]
indeed, it is enough to prove the following

**Lemma** for any nonnegative $k$ and $m$:

\[(q_0, a^{fk}a^m, X_2[F^k].\omega) \Rightarrow^* (q_0, a^m, \omega)\]

\[(q_0, a^{fk+1}a^m, X_1[F^k].\omega) \Rightarrow^*_\delta (q_0, a^m, \omega)\]
the case \( k = 0 \) is obvious assuming the above for \( k \), and any \( m \), consider \((q_0, a^{f_k+2}a^m, X_1[F^{k+1}].\omega)\) then
\[
(q_0, a^{f_k+2}a^m, X_1[F^{k+1}].\omega) \\
\vdash (q_1, a^{f_k+2}a^m, X_1[F^k].\omega) \\
\quad \delta(q_0, \epsilon, X_1 F) = (q_1, \text{pop}_2) \\
\vdash (q_0, a^{f_k+2}a^m, X_1[F^k].X_2[F^k].\omega) \\
\quad \delta(q_1, \epsilon, X_1 F) = (q_0, \text{push}_1(X_1 X_2)) \\
\vdash (q_0, a^{f_k}a^m, X_2[F^k].\omega) \Rightarrow^* (q_0, a^m, \omega) \\
\text{as } f_{k+2} = f_{k+1} + f_k \\
\text{and induction hypothesis}
2. tessellations in the hyperbolic plane
we shall see:

2.1 Poincaré’s disc model

2.2 the pentagrid and the heptagrid
2.1 Poincaré’s disc model

the disc
2.1 Poincaré’s disc model

a point $A$
2.1 Poincaré’s disc model

a point $A$

a line $\ell$
2.1 Poincaré’s disc model

A **secant** through $A$
which cuts $\ell$
2.1 Poincaré’s disc model

a parallel $p$

to $\ell$

through $A$
2.1 Poincaré’s disc model

another parallel $q$
to $\ell$
through $A$
2.1 Poincaré’s disc model

a non secant line $m$ to $\ell$ through $A$
2.1 Poincaré’s disc model

the **common perpendicular** to \( \ell \) and to \( m \)
a few useful properties

the sum of angles in a triangle:
   always less than $\pi$

non-secant lines:
   always a unique common perpendicular

no similarity in the Euclidean meaning
2.2 the pentagrid and the heptagrid

from Poincaré’s theorem:

ininitely many tilings of the hyperbolic plane by replicating a triangle in its side and, recursively, the images in their sides angles of the triangle, necessarily

\[
\frac{2\pi}{p}, \frac{2\pi}{q}, \frac{2\pi}{r},
\]

with \(p, q, r\) positive integers

satisfying:

\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1
\]
from now on focus on

the **pentagrid** and the **heptagrid**:

\(\{5,4\}\) \hspace{2cm} \(\{7,3\}\)
navigation in \{5,4\} and \{7,3\}

in the \textit{pentagrid}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{pentagrid}
\end{figure}

a quarter,
navigation in \( \{5,4\} \) and \( \{7,3\} \)

in the pentagrid

a quarter, and the underlying tree
navigation in \{5,4\} and \{7,3\}
in the heptagrid

a sector,
navigation in \{5,4\} and \{7,3\} in the heptagrid

a sector, and the underlying tree
navigation in \{5,4\} and \{7,3\}

in both cases:

a central tile and \(\alpha\) sectors, \(\alpha \in \{5, 7\}\)
the common tree for \(\{5,4\}\) and \(\{7,3\}\)

in both cases, the Fibonacci tree:
the common tree for \{5,4\} and \{7,3\}

why Fibonacci tree?
because
level \(n\) in the tree: \(f_{2n+1}\) nodes

two kinds of nodes: Black and White
with the rules:
\(B \rightarrow BW\), \(W \rightarrow BWW\)
hence the connection with 2-iterated pds
3. contour words of balls
we shall see:

3.1 balls in \{5, 4\} and \{7, 3\}

3.2 contour words and their recognition
### 3.1 balls in \{5, 4\} and in \{7, 3\}

common definition:

**path** between a tile \(A\) and a tile \(B\), \(A \neq B\):

sequence \(\{T_i\}_{0 \leq i \leq n}\) of tiles with:

- \(T_i\) and \(T_{i+1}\) share a side, \(i < n\),
- \(T_0 = A\), \(T_n = B\)
3.1 balls in \{5, 4\} and in \{7, 3\}

let \( \{T_i\}_{0 \leq i \leq n} \) be a path from \( A \) and \( B \)

\( n \) is the **length** of the path

now, \( d(A, B) \) is the length of a **shortest** path between \( A \) and \( B \),

\( A \neq B \),

\( d(A, B) = 0 \), when \( A = B \)
3.1 balls in \{5, 4\} and in \{7, 3\}

A ball of radius \(n\) around \(A\) is the set of tiles at a distance from \(A\) not greater than \(n\): denoted by \(B_A(n)\)
3.2 contour words and their recognition

contour of a ball:

the **contour** of the ball $B_A(n)$, denoted by $\partial B_A(n)$, is the set of tiles at a distance $n$ from $A$ exactly
3.2 contour words and their recognition

illustration:

pentagrid

heptagrid
3.2 contour words and their recognition

A contour word of $B_A(n)$ is a word $a^{s_n}$ where $s_n$ is the length of $\partial B_A(n)$.

Theorem (MM, arXiv 2009)

The language of the contour words of the balls of the pentagrid can be recognized by a 2-iterated pushdown automaton; the same for the heptagrid.
sketch of the proof:

each of the 5 sectors is spanned by a **Fibonacci tree**
the 2-iterated pushdown automaton performs a traversal of the tree
same proof for the heptagrid:

7 sectors here, each one spanned by the same **Fibonacci tree**
the 2-iterated pushdown automaton performs a traversal of the tree
two states: \( q_0 \) and \( q_1 \);
input word in \( \{b, w\}^* \); \( \Gamma = \{Z, B, W, F\} \);
initial state: \( q_0 \); initial stack: \( Z[\epsilon] \);
transition function \( \delta \), \( \alpha \in \{5, 7\} \):
\[
\delta(q_0, \epsilon, Z) = \{(q_0, \text{push}_2(F)), (q_0, \text{push}_1(W^\alpha))\}
\]
\[
\delta(q_0, \epsilon, ZF) = \{(q_0, \text{push}_2(FF)), (q_0, \text{push}_1(W^\alpha))\}
\]
\[
\delta(q_0, \epsilon, WF) = (q_1, \text{pop}_2)
\]
\[
\delta(q_0, \epsilon, BF) = (q_1, \text{pop}_2)
\]
\[
\delta(q_0, b, B) = (q_0, \text{pop}_1)
\]
\[
\delta(q_0, w, W) = (q_0, \text{pop}_1)
\]
\[
\delta(q_1, \epsilon, WF) = (q_0, \text{push}_1(BWW))
\]
\[
\delta(q_1, \epsilon, BF) = (q_0, \text{push}_1(BW))
\]
\[
\delta(q_1, \epsilon, W) = (q_0, \text{push}_1(BWW))
\]
\[
\delta(q_1, \epsilon, B) = (q_0, \text{push}_1(BW))
\]
the theorem can be extended to contour words of sectors:

pentagrid, see MM arXiv paper
similar extension to contour words of sectors:

heptagrid, see MM arXiv paper
the theorem can be extended
to balls and sectors

of the tilings \( \{p, 4\} \) of \( IH^2 \),
and the tilings \( \{p+2, 3\} \) of \( IH^2 \),
\( p \geq 5 \) in both cases

to balls

of the tiling \( \{5, 3, 4\} \) of \( IH^3 \)
and the tiling \( \{5, 3, 3, 4\} \) of \( IH^4 \)
4. in $3D$
we shall see:

4.1 \{5, 3, 4\}: the dodecagrid

4.2 recognition of contour words in the dodecagrid
4.1 \{5, 3, 4\}: the dodecagrid

first, extend Poincaé’s disc model:

the points: inside the \textbf{unit }\textit{nD ball}
the border, the \textbf{unit }\textit{n−1D ball}
    = the \textbf{points at infinity}

\textbf{hyperplanes}:

trace of diametral hyperplanes
or of \textit{n−1D balls}, orthogonal to the border

\textbf{k-hyperplanes}, \(1 \leq k \leq n−2\):

intersection of \(k+1\)-hyperplanes
only 4 tessellations in the hyperbolic 3D space

notation \( \{p, q, r\} \):

- \( p \) sides around a face
- \( q \) faces around a vertex
- \( r \) polyhedra around an edge

the tessellations:

\( \{3, 5, 3\} \quad \{4, 3, 5\} \quad \{5, 3, 4\} \quad \{5, 3, 5\} \)

the dodecagrid: \( \{5, 3, 4\} \)
an important tool: **Schlegel diagrams**

projection of the regular rectangular dodecahedron on the plane of a face
navigation in the dodecagrid

this gives us a \textbf{tree} in bijection with the tiling, whose generating rules are:

\[ O \rightarrow O^2H^6T, \quad H \rightarrow OH^6T, \quad T \rightarrow H^5T \]
a sector is the set of tiles generated by the tree $T$
we have 3 kinds of $\gamma$-sectors:
with $\gamma \in O-, H$- and $T$-sectors, depending on the root of the tree $T_{\gamma}$
a $k$-truncated $\gamma$-sector is generated by the first $k$ levels of $T_{\gamma}$
a ball $B_k$ is the union of 8 $k$-truncated $O$-sectors around a vertex
we can state:

**Theorem (MM, arxiv 2009)**

There is a 2-iterated pushdown automaton which recognizes the contour words of ball $B_k$, in the dodecagrid; for each $\gamma \in \{O, H, T\}$, there is a 2-iterated pushdown automaton which recognizes the contour word of any $k$-truncated $\gamma$-sector.
Thank you for your attention!