One-relation languages and $\omega$-code generators

Tran Vinh Duc, Igor Litovsky

I3S – Université de Nice - Sophia Antipolis – France

Journées Montoises d’Informatique Théorique
AMIENS, 2010
Infinite words

Let $A$ be a finite alphabet and $L$ be a language over $A$:

- An $\omega$-word is an infinite sequence of elements of $A$.
- An $\omega$-language is a set of $\omega$-words.
- The $\omega$-power of $L$ is defined by

$$L^\omega = \{u_0u_1\cdots | u_i \in L \setminus \epsilon\}.$$ 

Example

$(a^* b)^\omega$ is the set of $\omega$-words which contains an infinite number of occurrences of $b$. 
Family of generators

The family of generators of $L^\omega$

$$[L^\omega] = \{G \subseteq A^* \mid G^\omega = L^\omega\}$$

Example

• $L^\omega = (a^* b)^\omega$ is the set of $\omega$-words which contains an infinite number of occurrences of $b$.
• $G = a^* ba^*$ is another generator of $L^\omega$,

$$(a^* b)^\omega = (a^* ba^*)^\omega.$$
Côdes

- A langage $L$ is a **code** if any word in $A^*$ has *at most* one factorization in words of $L$.
- A language $L$ is an $\omega$-**code** if any $\omega$-word in $A^\omega$ has *at most* one factorization in words of $L$.

**Example**

- $\{a, ab, ba\}$ is not a code
  
  $$aba = (a)ba = (ab)a$$

- $\{a, ab, b^2\}$ is a code but not an $\omega$-code
  
  $$abbb \cdots = (a)b^2b^2 \cdots = (ab)b^2b^2 \cdots$$
An open problem

“Does a rational $\omega$-language have an $\omega$-code generator?”

Example

The language $a \cup (ab)^*ba$ is an $\omega$-code generator of $\{a, ab, ba\}^\omega$.

The problem is solved for prefix code generators [Litovsky, 91].
Finite vs infinite case

$L \neq \emptyset$ and $L$ rational.

<table>
<thead>
<tr>
<th></th>
<th>maximal generators</th>
<th>minimal generators</th>
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<tbody>
<tr>
<td>monoid $L^*$</td>
<td>$L^*$</td>
<td>$(L^* \setminus \epsilon) \setminus (L^* \setminus \epsilon)^2$</td>
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<tr>
<td>$\omega$-power $L^\omega$</td>
<td>finite number</td>
<td>?</td>
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- $L^*$ is generated by a code iff its minimal generator is a code.
- For $\omega$-power, it is very different:

Example

$L = \{a^2, a^3, b\}$ is a minimal generator of $L^\omega$.
$L^*$ is the greatest generator of $L^\omega$.
$\{a^2, a^3b, b\}$ is an $\omega$-code generator of $L^\omega$. 
Remarks

1) If $L$ is a two-element language, then $L^\omega$ always has an $\omega$-code generator.
   • If $L = \{u, v\}$ is not an $\omega$-code, then $u, v$ are powers of a same word. Thus $\{u, v\}^\omega = u^\omega$.

2) There exists a three-element language $L$ such that $L^\omega$ has no $\omega$-code generator.

Example

$\{a, ab, baba\}^\omega$ has no $\omega$-code generator.
Zero-relation languages $\equiv$ Codes

Theorem (Julia et al., 1996)

Let $L$ be a code such that $L^*$ is the greatest generator of $L^\omega$. Then $L^\omega$ has an $\omega$-code generator if and only if $L$ is an $\omega$-code generator.

What about one-relation languages?
Non-trivial Relations

\[ L = \{u_1, u_2, \ldots, u_n\} \text{ a finite language over } A; \]
\[ \Sigma = \{x_1, x_2, \ldots, x_n\} \text{ an alphabet.} \]

A relabeling of \( L \) is a morphism \( \sim : \Sigma^\infty \to L^\infty \) defined by

\[ x_i \mapsto u_i \]

The set of \textit{relations} is satisfied by \( L \):

\[ \equiv = \{ (x, y) \in \Sigma^\infty \times \Sigma^\infty \mid \tilde{x} = \tilde{y} \} \]
Example

For $L = \{a, ab, ba\}$ and $\Sigma = \{0, 1, 2\}$. 

02 \cong 10 

022 \cong 102
For $L = \{a, ab, ba\}$ and $\Sigma = \{0, 1, 2\}$.

The relation $x \equiv y$ is \textbf{minimal} if $x' \not\equiv y'$ for all $x' \in \text{Pref}(x)$ and $y' \in \text{Pref}(y)$.

The set of minimal relations of $L$ is

\[
\begin{cases}
02^n \equiv 1^n0 & \text{for all } n > 0 \\
02^\omega \equiv 1^\omega
\end{cases}
\]
Example

For \( L = \{a, ab, ba\} \) and \( \Sigma = \{0, 1, 2\} \), the set of minimal relations of \( L \) is recognized by a finite automaton:

Figure: Automaton recognizing the language \((0, 1)(2, 1)\infty(2, 0)\).
Example

For \( L = \{a, ab, ba\} \) and \( \Sigma = \{0, 1, 2\} \), the set of minimal relations of \( L \) is recognized by a finite automaton:

![Automaton diagram]

**Figure:** Automaton recognizing the language \((0, 1)(2, 1)\infty(2, 0)\).

Thus, the set of minimal relations of \( L \) is

\[
\left\{ \begin{align*}
02^n & \equiv 1^n0 \quad \text{for all } n > 0 \\
02^\omega & \equiv 1^\omega 
\end{align*} \right. 
\]
Properties of set \( \cong \)

P1) \( \cong \) is a congruence;
P2) If \( xw \cong w \) then \( x^\omega \cong w \).
Properties of set $\cong$

P1) $\cong$ is a congruence;

P2) If $xw \cong w$ then $x^\omega \cong w$.

Some relations are derivated from the others:

Example

$$02 \cong 10 \Rightarrow 022 \cong 102 \cong 110$$
$$\Rightarrow 0222 \cong 1022 \cong 1102 \cong 1110$$
$$\ldots$$
$$\Rightarrow 02^n \cong 1^n0.$$ 

$$10 \cong 02 \Rightarrow 102^\omega \cong 022^\omega = 02^\omega$$
$$\Rightarrow 1^\omega \cong 02^\omega.$$
One-relation languages

Definition
We say that $L$ is a one-relation language if there is a pair $(x, y) \in \Sigma^+ \times \Sigma^+$, $x \neq y$ such that $\sim$ is the smallest subset of $\Sigma^\infty \times \Sigma^\infty$ containing $(x, y)$ and satisfying the properties P1, P2. Then $x \equiv y$ is called basic relation.
One-relation languages

Definition
We say that $L$ is a one-relation language if there is a pair $(x, y) \in \Sigma^+ \times \Sigma^+$, $x \neq y$ such that $\sim$ is the smallest subset of $\Sigma^\infty \times \Sigma^\infty$ containing $(x, y)$ and satisfying the properties P1, P2. Then $x \sim y$ is called basic relation.

Example

- Let $L = \{a, ab, ba\}$, and $\Sigma = \{0, 1, 2\}$. $L$ is a one-relation language, where $02 \sim 10$ is the basic relation.
- Let $L = \{a, ab, ba, ac, ca\}$, and $\Sigma = \{0, 1, 2, 3, 4\}$. $L$ is not one-relation language; i.e., we have $02 \sim 10$ and $04 \sim 30$. 
Overlaps of basic relation

\[ OVL(x, y) = (\text{Pref}(x) \cap \text{Suf}(y)) \cup (\text{Suf}(x) \cap \text{Pref}(y)) \].

The overlapping words in the basic relation generates other relations.

Example

- Let \( L = \{a, a^2b, ba^2\} \) and \( \Sigma = \{0, 1, 2\} \).
- The basic relation 002 \( \cong 100 \), and \( OVL(002, 100) = \{0, 00\} \).
The set of overlapping words is empty

Proposition

Let $L$ be a finite language such that $L^*$ is the greatest generator of $L^\omega$. If $L$ is one-relation language with the basic relation $x \preceq y$ and $OVL(x, y) = \emptyset$, then $L^\omega$ has no $\omega$-code generator.
Proposition
Let $L$ be a finite language such that $L^*$ is the greatest generator of $L^\omega$. If $L$ is one-relation language with the basic relation $x \equiv y$ and $OVL(x, y) = \emptyset$, then $L^\omega$ has no $\omega$-code generator.

Example
Let $L = \{a, ab, bc, c\}$ and $\Sigma = \{0, 1, 2, 3\}$. $L$ is a one-relation language, where $02 \equiv 13$ is the basic relation.
The set of overlapping words is singleton

Proposition

Let $L$ be a finite language such that $L^*$ is the greatest generator of $L^\omega$. If $L$ is a one-relation language with the basic relation $xz \cong yx$ and $OVL(xz, yx) = \{x\}$, then $L^\omega$ has an $\omega$-code generator if and only if

\[ x = 0 \text{ and } y = 1 \]

or

\[ x = 0, z = 2, \text{ and } y \in 10^+. \]
The set of overlapping words is singleton

Proposition

Let $L$ be a finite language such that $L^*$ is the greatest generator of $L^\omega$. If $L$ is a one-relation language with the basic relation $xz \cong yx$ and $OVL(xz, yx) = \{x\}$, then $L^\omega$ has an $\omega$-code generator if and only if

$$x = 0 \text{ and } y = 1$$

or

$$x = 0, z = 2, \text{ and } y \in 10^+.$$

Example

Let $L = \{a, ab, ba\}$, and $\Sigma = \{0, 1, 2\}$. $L$ is a one-relation language, where $02 \cong 10$ is the basic relation. We have an $\omega$-code generator $\widetilde{G}$, where $G = 0 + 1^*2$. 
Proposition

Let $L$ be a finite language such that $L^*$ is the greatest generator of $L^\omega$. If $L$ is a one-relation language with the basic relation $xz \simeq yx$ and $OVL(xz, yx) = \{x\}$, then $L^\omega$ has no finite $\omega$-code generator.
Conclusions: some natural extensions

Example

- \( L = \{ a, a^2b, ba^2 \} \) and \( \Sigma = \{ 0, 1, 2 \} \).
- Basic relation \( 002 \equiv 100 \) and \( OVL(002, 100) = \{ 0, 00 \} \).
- \( \omega \)-code generator of \( L^\omega \):
  \[
a \cup \{ a^2b, a^2ba \}^*ba^2.
  \]

Example

- \( L = \{ a, ab, ba, ac, ca \} \), and \( \Sigma = \{ 0, 1, 2, 3, 4 \} \);
- Basic relations: \( 02 \equiv 10 \) and \( 04 \equiv 30 \);
- \( \omega \)-code generator of \( L^\omega \):
  \[
a \cup \{ ab, ac \}^*\{ ba, ca \}.
  \]